

# 第8章 超ラグランジアン#2

## Non-abelian カイラルゲージ場

カイラルゲージ超場は

$$\begin{aligned}
 W_\alpha(y) &= \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta G_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x)) (D_\alpha \exp(V(x))) \\
 &\Leftarrow V_{WS} = \theta \sigma^\mu \bar{\theta} V_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta} D \\
 W_\alpha(y) &= \frac{1}{2g} \bar{D}^2 \left( D_\alpha V_{WS} - \frac{1}{2} [V_{WS}, D_\alpha V_{WS}] \right) \\
 &= \left( 4i\lambda_\alpha^a(y) - \left[ 4D^a(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_\alpha^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \right) t^a \\
 &\Leftarrow \mathcal{D}_\mu = \partial_\mu + \frac{1}{2} iV_\mu = \partial_\mu + ig \sum_{a=1}^N V_\mu^a t^a, \quad V_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - g f^{abc} V_\mu^b V_\nu^c
 \end{aligned}$$

で表せるので、超対称性不変なラグランジアンになるのは、

$$\mathcal{L} \supset \text{Tr} \left( W^\alpha(y, \theta) W_\alpha(y, \theta) \right) \Big|_F + \text{h.c.} = \int \left( \text{Tr} \left( W^\alpha(y, \theta) W_\alpha(y, \theta) \right) + \text{h.c.} \right) d^2\theta$$

である。そこで、

$$W_\alpha(y, \theta) \equiv W_\alpha^a(y, \theta) t^a = \left( 4i\lambda_\alpha^a(y) - \left[ 4D^a(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_\alpha^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \right) t^a$$

として、

$$\begin{aligned}
 W_\alpha^a &= 4i\lambda_\alpha^a(y) - \left[ 4D^a(y) \theta_\alpha + 2(\sigma^{\mu\nu} \theta)_\alpha V_{\mu\nu}^a(y) \right] + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \\
 \Rightarrow \varepsilon^\alpha W_\alpha^a &= 4i\varepsilon\lambda^a(y) - \left[ 4\varepsilon\theta D^a(y) + 2(\varepsilon\sigma^{\mu\nu} \theta) V_{\mu\nu}^a(y) \right] + 4(\varepsilon\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \\
 \Rightarrow W^{\alpha\alpha} \varepsilon_\alpha &= 4i\lambda^a(y) \varepsilon - \left[ 4\theta\varepsilon D^a(y) - 2(\theta\sigma^{\mu\nu} \varepsilon) V_{\mu\nu}^a(y) \right] - 4(\mathcal{D}_\mu \bar{\lambda}(y) \bar{\sigma}^\mu \varepsilon)^\alpha \theta\theta \\
 \Rightarrow W^{\alpha\alpha} &= 4i\lambda^{\alpha\alpha}(y) - \left[ 4\theta^\alpha D^a(y) - 2(\theta\sigma^{\mu\nu})^\alpha V_{\mu\nu}^a(y) \right] - 4(\mathcal{D}_\mu \bar{\lambda}(y) \bar{\sigma}^\mu)^{\alpha\alpha} \theta\theta
 \end{aligned}$$

に注意すると、

$$\begin{aligned}
 \text{Tr} \left( W^\alpha(y, \theta) W_\alpha(y, \theta) \right) &= \text{Tr} \left( W^{\alpha\alpha}(y, \theta) t^a W_\alpha^b(y, \theta) t^b \right) \Leftarrow \text{Tr} \left( t^a t^b \right) = \delta^{ab} T(R) \quad \text{with } T(\text{FUND}) = 1/2 \\
 &= T(R) W^{\alpha\alpha}(y, \theta) W_\alpha^a(y, \theta)
 \end{aligned}$$

なので、

$$\mathcal{L} \supset \text{Tr} \left( W^\alpha(y, \theta) W_\alpha(y, \theta) \right) \Big|_F + \text{h.c.} = T(R) \left( W^{\alpha\alpha}(y, \theta) W_\alpha^a(y, \theta) \Big|_F + \text{h.c.} \right)$$

になる。この  $F$  項は

$$\begin{aligned}
 W^{\alpha\alpha}(y, \theta) W_\alpha^a(y, \theta) &= \left( 4i\lambda^{\alpha\alpha}(y) - \left[ 4\theta^\alpha D^a(y) - 2(\theta\sigma^{\mu\nu})^\alpha V_{\mu\nu}^a(y) \right] - 4(\mathcal{D}_\mu \bar{\lambda}(y) \bar{\sigma}^\mu)^{\alpha\alpha} \theta\theta \right) \\
 &\quad \cdot \left( 4i\lambda_\alpha^a(y) - \left[ 4D^a(y) \theta_\alpha + 2(\sigma^{\mu\nu} \theta)_\alpha V_{\mu\nu}^a(y) \right] + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \right)
 \end{aligned}$$

より、

$$\begin{aligned}
W^{a\alpha}(y, \theta)W_{\alpha}^a(y, \theta) &= (4i\lambda^{a\alpha}(y)) \left( 4(\sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(y))_{\alpha}^a \theta \theta \right) \\
&+ \left( -\left[ 4\theta^{\alpha} D^a(y) - 2(\theta \sigma^{\mu\nu})^{\alpha} V_{\mu\nu}^a(y) \right] \right) \left( -\left[ 4D^a(y) \theta_{\alpha} + 2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(y) \right] \right) \\
&+ \left( -4(\mathcal{D}_{\mu} \bar{\lambda}(y) \bar{\sigma}^{\mu})^{a\alpha} \right) (4i\lambda_{\alpha}^a(y)) + (\dots) \Leftarrow y \rightarrow x \text{ because of } \theta \sigma \bar{\theta} \theta = 0 \\
&= 16i\lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) \theta \theta - 16i \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \theta \theta \\
&+ \theta^{\beta'} \theta_{\beta} \left( -\left[ 4D^a(x) \delta_{\beta'}^{\alpha} - 2(\sigma^{\mu\nu})_{\beta'}^{\alpha} V_{\mu\nu}^a(x) \right] \right) \left( -\left[ 4D^a(x) \delta_{\alpha}^{\beta} + 2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x) \right] \right) + (\dots) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta \\
&+ \frac{1}{2} \theta \theta \delta_{\beta}^{\beta'} \left( -\left[ 4D^a(x) \delta_{\beta'}^{\alpha} - 2(\sigma^{\mu\nu})_{\beta'}^{\alpha} V_{\mu\nu}^a(x) \right] \right) \left( -\left[ 4D^a(x) \delta_{\alpha}^{\beta} + 2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x) \right] \right) + (\dots) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta \\
&+ \frac{1}{2} \theta \theta \left( -\left[ 4D^a(x) \delta_{\beta}^{\alpha} - 2(\sigma^{\mu\nu})_{\beta}^{\alpha} V_{\mu\nu}^a(x) \right] \right) \left( -\left[ 4D^a(x) \delta_{\alpha}^{\beta} + 2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x) \right] \right) + (\dots) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta \\
&+ \frac{1}{2} \theta \theta (-4D^a(x) \delta_{\beta}^{\alpha}) (-4D^a(x) \delta_{\alpha}^{\beta}) + \frac{1}{2} \theta \theta (-4D^a(x) \delta_{\beta}^{\alpha}) (-2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x)) \\
&+ \frac{1}{2} \theta \theta \left( -\left[ -2(\sigma^{\mu\nu})_{\beta}^{\alpha} V_{\mu\nu}^a(x) \right] \right) (-4D^a(x) \delta_{\alpha}^{\beta}) + \frac{1}{2} \theta \theta \left( -\left[ -2(\sigma^{\mu\nu})_{\beta}^{\alpha} V_{\mu\nu}^a(x) \right] \right) (-2(\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x)) + (\dots) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta + \frac{1}{2} 16\theta \theta 2D^{a2}(x) + \frac{1}{2} 8\theta \theta D^a(x) \text{Tr}(\sigma^{\rho\tau}) V_{\rho\tau}^a(x) \\
&- \frac{1}{2} 8\theta \theta \text{Tr}(\sigma^{\mu\nu}) V_{\mu\nu}^a(x) D^a(x) - \frac{1}{2} 4\theta \theta (\sigma^{\mu\nu})_{\beta}^{\alpha} V_{\mu\nu}^a(x) (\sigma^{\rho\tau})_{\alpha}^{\beta} V_{\rho\tau}^a(x) + (\dots) \\
&\Leftarrow \text{Tr}(\sigma^{\mu\nu}) = 0, \quad (\sigma^{\mu\nu})_{\beta}^{\alpha} (\sigma^{\rho\tau})_{\alpha}^{\beta} = \text{Tr}(\sigma^{\mu\nu} \sigma^{\rho\tau}) = 2(g^{\mu\rho} g^{\nu\tau} - g^{\nu\rho} g^{\mu\tau} + i\varepsilon^{\mu\nu\rho\tau}) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta + 16\theta \theta D^{a2}(x) \\
&- \frac{1}{2} 4\theta \theta 2(g^{\mu\rho} g^{\nu\tau} - g^{\nu\rho} g^{\mu\tau} + i\varepsilon^{\mu\nu\rho\sigma}) V_{\mu\nu}^a(x) V_{\rho\tau}^a(x) + (\dots) \\
&= 16i \left( \lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) - \mathcal{D}_{\mu} \bar{\lambda}(x) \bar{\sigma}^{\mu} \lambda(x) \right) \theta \theta + 16\theta \theta D^{a2}(x) - 8\theta \theta V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \\
&- 4i\varepsilon^{\mu\nu\rho\tau} \theta \theta V_{\mu\nu}^a(x) V_{\rho\tau}^a(x) + (\dots) \\
&= 32i\lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) \theta \theta + 16\theta \theta D^{a2}(x) - 8\theta \theta V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - 4i\varepsilon^{\mu\nu\rho\tau} \theta \theta V_{\mu\nu}^a(x) V_{\rho\tau}^a(x) + (\dots)
\end{aligned}$$

故に

$$\begin{aligned}
W^{a\alpha}(y, \theta)W_{\alpha}^a(y, \theta) \Big|_F &= 32i\lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) + 16D^{a2}(x) - 8V^{a\mu\nu}(x) V_{\mu\nu}^a(x) - 4i\varepsilon^{\mu\nu\rho\tau} \theta \theta V_{\mu\nu}^a(x) V_{\rho\tau}^a(x) \\
&= 32 \left\{ i\lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\mu\nu}^a(x) - \frac{1}{8} i\varepsilon^{\mu\nu\rho\tau} V_{\mu\nu}^a(x) V_{\rho\tau}^a(x) \right\} \\
&= 32 \left\{ i\lambda(x) \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\mu\nu}^a(x) - \frac{1}{4} i\tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \right\} \Leftarrow \tilde{V}^{a\mu\nu} = \frac{1}{2} i\varepsilon^{\mu\nu\rho\tau} V_{\rho\tau}^a
\end{aligned}$$

一方、

$$\begin{aligned}
\bar{W}_\alpha^a(y, \theta) \bar{W}^{\alpha a}(y, \theta) \Big|_F &= 32 \left\{ -i(\mathcal{D}_\mu \bar{\lambda}(x))^\dagger \sigma^\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right\} \\
&\Leftarrow (\mathcal{D}_\mu \bar{\lambda}(x))^\dagger = \left( \left( \partial_\mu + ig \sum_{a=1}^N V_\mu^a t^a \right) \bar{\lambda}(x) \right)^\dagger = \partial_\mu \lambda(x) - ig \lambda(x) \sum_{a=1}^N V_\mu^a t^a \\
&= 32 \left\{ -i \left( \partial_\mu \lambda(x) - ig \lambda(x) \sum_{a=1}^N V_\mu^a t^a \right) \sigma^\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right\} \\
&= 32 \left\{ \begin{aligned} &-i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) + i\lambda(x) \sigma^\mu \partial_\mu \bar{\lambda}(x) - i \left( -ig \lambda(x) \sum_{a=1}^N V_\mu^a t^a \right) \sigma^\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) \\ &-\frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \end{aligned} \right\} \\
&= 32 \left\{ -i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) + i\lambda(x) \sigma^\mu \left( \partial_\mu + ig \sum_{a=1}^N V_\mu^a t^a \right) \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right\} \\
&= 32 \left\{ -i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) + i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right\} \\
&= W^{\alpha a}(y, \theta) W_\alpha^a(y, \theta) \Big|_F - 32i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x))
\end{aligned}$$

より、

$$\begin{aligned}
\mathcal{L} \supset \text{Tr}(W^a(y, \theta) W_\alpha(y, \theta)) + \text{h.c.} \Big|_F &= T(R) \left( W^{\alpha a}(y, \theta) W_\alpha^a(y, \theta) \Big|_F + \text{h.c.} \right) \Big|_F \\
&= T(R) \left( W^{\alpha a}(y, \theta) W_\alpha^a(y, \theta) \Big|_F + W^{\alpha a}(y, \theta) W_\alpha^a(y, \theta) \Big|_F - 32i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) + \text{h.c.} \right) \\
&= 2T(R) \left( W^{\alpha a}(y, \theta) W_\alpha^a(y, \theta) \Big|_F - 16i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) \right) \\
&= 2T(R) \left( 32 \left\{ i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - \frac{1}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \right\} - 16i(\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) \right) \\
&= 64T(R) \left( i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - \frac{1}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) - \frac{i}{2} (\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) \right)
\end{aligned}$$

つまり、

$$\begin{aligned}
L &= \frac{1}{64T(R)} \int \left\{ \text{Tr}(W^a(y, \theta) W_\alpha(y, \theta)) + \text{h.c.} \right\} \Big|_F d^4x \\
&= \int \left( i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - \frac{1}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) - \frac{i}{2} (\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)) \right) d^4x \\
&= \int \left( i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - \frac{1}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \right) d^4x \Leftarrow \tilde{V}^{a\mu\nu} = \frac{1}{2} i\epsilon^{\mu\nu\rho\tau} V_{\rho\tau}^a
\end{aligned}$$

になり、ゲージ超場の相互作用を与える。 $V\tilde{V}$ 項はインスタントンの影響があると寄与し、

$$\text{Im} \left[ \left( \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \right) \frac{1}{4\pi} L \right] = \frac{1}{4\pi} \text{Im}(\tau L) = \int \left( \begin{aligned} &\frac{1}{g^2} \left( i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right) \\ &-\frac{\theta}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \end{aligned} \right) d^4x$$

と構成される。以上をまとめると、

$$\begin{aligned}
\text{Lagrangian} &= \frac{1}{4\pi} \text{Im}(\tau L) \text{ with } \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \\
&= \int \left( \frac{1}{g^2} \left( i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) \right) - \frac{\theta}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \right) d^4x \\
L &= \frac{1}{64T(R)} \int \left\{ \text{Tr}(W^\alpha(y, \theta) W_\alpha(y, \theta)) + \text{h.c.} \right\} \Big|_F d^4x \equiv \int \mathcal{L}(x) d^4x \\
&\text{with } \text{Tr}(t^a t^b) = \delta^{ab} T(R) \text{ \& } T(\text{FUND}) = 1/2 \\
\mathcal{L}(x) &= i\lambda(x) \sigma^\mu \mathcal{D}_\mu \bar{\lambda}(x) + \frac{1}{2} D^{a2}(x) - \frac{1}{4} V^{a\mu\nu}(x) V_{\rho\tau}^a(x) - \frac{1}{4} \tilde{V}^{a\mu\nu}(x) V_{\mu\nu}^a(x) \\
&\text{with } \mathcal{D}_\mu = \partial_\mu + \frac{1}{2} i V_\mu = \partial_\mu + ig \sum_{a=1}^N V_\mu^a t^a, \quad V_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - gf^{abc} V_\mu^b V_\nu^c \\
W_\alpha(y) &= \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta G_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x)) (D_\alpha \exp(V(x))) \\
V_{ws} &= \theta \sigma^\mu \bar{\theta} V_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta} D \\
W_\alpha(y) &\equiv W_\alpha^a(y) t^a = \frac{1}{2g} \bar{D}^2 \left( D_\alpha V_{ws} - \frac{1}{2} [V_{ws}, D_\alpha V_{ws}] \right) \\
&= \left( 4i\lambda_\alpha^a(y) - \left[ 4D^a(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}(y))_\alpha^a \theta\theta \right) t^a
\end{aligned}$$

## ゲージ相互作用

ゲージ変換での不変性は「第5章 超ゲージ対称性」での議論のように

$$\Phi^\dagger(x) \exp(2gV^a t^a) \Phi(x)$$

の不変性を出発点にしている。その結果、

$$\exp(V') = \exp(-i\Lambda^\dagger) \exp(V) \exp(i\Lambda) \text{ with } V = 2g \sum_{a=1}^N V^a t^a \text{ \& } \Lambda = 2g \sum_{a=1}^N \Lambda^a t^a$$

および、

$$\delta V = V' - V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2} [V, \Lambda + \Lambda^\dagger] + \frac{1}{12} i [[\Lambda - \Lambda^\dagger, V], V] + (\dots)$$

が要請された。さて、特別なゲージ (Wess-Zumino ゲージ) に固定できるので、それを、

$$V_{ws} = \theta \sigma^\mu \bar{\theta} V_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta} D$$

で表す。そこで、このゲージ相互作用を成分場を使用して表す。Wess-Zumino ゲージでは  $VVV = 0$ 。

$$\Phi^\dagger(x) \exp(V) \Phi(x) \Big|_D = \Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D + \Phi^\dagger(y, \theta) V \Phi(y, \theta) \Big|_D + \frac{1}{2} \Phi^\dagger(y, \theta) VV \Phi(y, \theta) \Big|_D$$

ここで、**第一項**は、

$$\Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D = \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left( -\frac{\partial \bar{\psi}(x)}{\partial x^\mu} \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \frac{\partial \psi(x)}{\partial x^\mu} \right) + |F(x)|^2$$

である。第二項は、

$$\begin{aligned}
\Phi^\dagger(y, \theta) V \Phi(y, \theta) &= \Phi^\dagger(y, \theta) \left( \theta \sigma^\mu \bar{\theta} V_\mu + i \theta \theta \bar{\theta} \bar{\lambda} - i \bar{\theta} \theta \theta \lambda + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} D \right) \Phi(y, \theta) \\
&= \left( \phi^\dagger(y) - \sqrt{2} \bar{\theta} \bar{\psi}(y) + F^\dagger(y) \bar{\theta} \bar{\theta} \right) \theta \sigma^\mu \bar{\theta} V_\mu \left( \phi(y) - \sqrt{2} \theta \psi(y) + F(y) \theta \theta \right) \\
&\quad + \left[ \left( \phi^\dagger(y) - \sqrt{2} \bar{\theta} \bar{\psi}(y) + F^\dagger(y) \bar{\theta} \bar{\theta} \right) i \theta \theta \bar{\theta} \bar{\lambda} \left( \phi(y) - \sqrt{2} \theta \psi(y) + F(y) \theta \theta \right) + \text{h.c.} \right] \\
&\quad + \frac{1}{2} \left( \phi^\dagger(y) - \sqrt{2} \bar{\theta} \bar{\psi}(y) + F^\dagger(y) \bar{\theta} \bar{\theta} \right) \theta \theta \bar{\theta} \bar{\theta} D \left( \phi(y) - \sqrt{2} \theta \psi(y) + F(y) \theta \theta \right) \\
&= \left[ \left( \phi^\dagger(x) \right) \theta \sigma^\mu \bar{\theta} V_\mu \left( -i \theta \sigma^\nu \bar{\theta} \partial_\nu \phi(x) \right) + \text{h.c.} \right] \\
&\quad + \left( -\sqrt{2} \bar{\theta} \bar{\psi}(x) \right) \theta \sigma^\mu \bar{\theta} V_\mu \left( -\sqrt{2} \theta \psi(x) \right) + \left[ \left( -\sqrt{2} \bar{\theta} \bar{\psi}(x) \right) i \theta \theta \bar{\theta} \bar{\lambda} \phi(x) + \text{h.c.} \right] + \frac{1}{2} \phi^\dagger(x) \theta \theta \bar{\theta} \bar{\theta} D \phi(x) + (\dots) \\
&\quad \Leftarrow \bar{\theta} \bar{\psi}(x) \theta \bar{\lambda} = \bar{\theta}_\alpha \bar{\psi}^\alpha(x) \bar{\lambda}_\beta \bar{\theta}^\beta = -\frac{1}{2} \bar{\theta} \bar{\theta} \bar{\psi}(x) \bar{\lambda}, \quad (\theta \sigma^\nu \bar{\theta})(\theta \sigma^\mu \bar{\theta}) = \frac{1}{2} g^{\mu\nu} \theta \theta \bar{\theta} \bar{\theta} \\
&\quad \Leftarrow \bar{\theta} \bar{\psi} \theta \sigma^\mu \bar{\theta} V_\mu \cdot \theta \psi = \bar{\theta}_\alpha \bar{\psi}^\alpha (\theta \sigma^\mu)_\beta \bar{\theta}^\beta V_\mu \cdot \theta \psi = -\frac{1}{2} \bar{\theta} \bar{\theta} \cdot \theta \sigma^\mu \bar{\psi} V_\mu \cdot \theta \psi = \frac{1}{2} \bar{\theta} \bar{\theta} \cdot \bar{\psi} \bar{\sigma}^\mu \theta V_\mu \cdot \theta \psi \\
&\quad = \frac{1}{2} \bar{\theta} \bar{\theta} \cdot (\bar{\psi} \bar{\sigma}^\mu)^\alpha \theta_\alpha V_\mu \cdot \theta^\beta \psi_\beta = -\frac{1}{4} \bar{\psi} \bar{\sigma}^\mu V_\mu \psi \theta \theta \bar{\theta} \bar{\theta} \\
&= \left[ \frac{1}{2} g^{\mu\nu} \theta \theta \bar{\theta} \bar{\theta} \phi^\dagger(x) V_\mu \left( -i \partial_\nu \phi(x) \right) + \text{h.c.} \right] + \frac{1}{2} \phi^\dagger(x) D \phi(x) \theta \theta \bar{\theta} \bar{\theta} \\
&\quad + \left( -\sqrt{2} \right) \left( -\sqrt{2} \right) \left( -\frac{1}{4} \bar{\psi} \bar{\sigma}^\mu V_\mu \psi \right) \theta \theta \bar{\theta} \bar{\theta} + \left[ \left( -\sqrt{2} \bar{\theta} \bar{\psi}(x) \right) i \theta \theta \bar{\theta} \bar{\lambda} \phi(x) + \text{h.c.} \right] + (\dots) \\
&= -i \frac{1}{2} \left( \phi^\dagger(x) V^\mu \partial_\mu \phi(x) + \text{h.c.} \right) \theta \theta \bar{\theta} \bar{\theta} + \frac{1}{2} \phi^\dagger(x) D \phi(x) \theta \theta \bar{\theta} \bar{\theta} \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) \theta \theta \bar{\theta} \bar{\theta} - \sqrt{2} i \left[ -\frac{1}{2} \bar{\theta} \bar{\theta} \bar{\psi}(x) \bar{\lambda} \phi(x) \theta \theta + \text{h.c.} \right] + (\dots) \\
&= -\frac{1}{2} \left( i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) + \text{h.c.} \right) \theta \theta \bar{\theta} \bar{\theta} + \frac{1}{2} \phi^\dagger(x) D \phi(x) \theta \theta \bar{\theta} \bar{\theta} \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) \theta \theta \bar{\theta} \bar{\theta} + \frac{1}{\sqrt{2}} \left( i \bar{\psi}(x) \bar{\lambda} \phi(x) + \text{h.c.} \right) \theta \theta \bar{\theta} \bar{\theta} + (\dots) \\
&= -\frac{1}{2} \left( i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) - i \partial_\mu \phi^\dagger(x) V^\mu \phi(x) \right) \theta \theta \bar{\theta} \bar{\theta} + \frac{1}{2} \phi^\dagger(x) \theta \theta \bar{\theta} \bar{\theta} D \phi(x) \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) \theta \theta \bar{\theta} \bar{\theta} + \frac{1}{\sqrt{2}} \left( i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x) \right) \theta \theta \bar{\theta} \bar{\theta} + (\dots)
\end{aligned}$$

故に、

$$\begin{aligned}
\Phi^\dagger(y, \theta) V \Phi(y, \theta) \Big|_D &= -\frac{1}{2} \left( i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) - i \partial_\mu \phi^\dagger(x) V^\mu \phi(x) \right) + \frac{1}{2} \phi^\dagger(x) D \phi(x) \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) + \frac{1}{\sqrt{2}} \left( i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x) \right)
\end{aligned}$$

第三項は、

$$\begin{aligned}
\Phi^\dagger(y, \theta) V V \Phi(y, \theta) &= \Phi^\dagger(y, \theta) \left( \theta \sigma^\mu \bar{\theta} V_\mu + i \theta \theta \bar{\theta} \bar{\lambda} - i \bar{\theta} \theta \theta \lambda + \theta \theta \bar{\theta} \bar{\theta} D / 2 \right) \\
&\quad \cdot \left( \theta \sigma^\nu \bar{\theta} V_\nu + i \theta \theta \bar{\theta} \bar{\lambda} - i \bar{\theta} \theta \theta \lambda + \theta \theta \bar{\theta} \bar{\theta} D / 2 \right) \Phi(y, \theta)
\end{aligned}$$

$$\begin{aligned}
&= \phi^\dagger(x) \theta \sigma^\mu \bar{\theta} V_\mu \theta \sigma^\nu \bar{\theta} V_\nu \phi(x) + (\dots) \Leftarrow (\theta \sigma^\nu \bar{\theta})(\theta \sigma^\mu \bar{\theta}) = \frac{1}{2} g^{\mu\nu} \theta \theta \bar{\theta} \bar{\theta} \\
&= \phi^\dagger(x) \frac{1}{2} g^{\mu\nu} \theta \theta \bar{\theta} \bar{\theta} V_\mu V_\nu \phi(x) + (\dots)
\end{aligned}$$

故に、

$$\Phi^\dagger(y, \theta) V V \Phi(y, \theta) \Big|_D = \frac{1}{2} \phi^\dagger(x) V^\mu V_\mu \phi(x)$$

である。

以上から、

$$\begin{aligned}
\Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D &= \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left( \psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \\
\Phi^\dagger(y, \theta) V \Phi(y, \theta) \Big|_D &= -\frac{1}{2} (i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) - i \partial_\mu \phi^\dagger(x) V^\mu \phi(x)) + \frac{1}{2} \phi^\dagger(x) D \phi(x) \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) + \frac{1}{\sqrt{2}} (i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x)) \\
\Phi^\dagger(y, \theta) V V \Phi(y, \theta) \Big|_D &= \frac{1}{2} \phi^\dagger(x) V^\mu V_\mu \phi(x)
\end{aligned}$$

求める相互作用は、

$$\begin{aligned}
\Phi^\dagger(x) \exp(V) \Phi(x) \Big|_D &= \Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D + \Phi^\dagger(y, \theta) V \Phi(y, \theta) \Big|_D + \frac{1}{2} \Phi^\dagger(y, \theta) V V \Phi(y, \theta) \Big|_D \\
&= \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left( -\frac{\partial \bar{\psi}(x)}{\partial x^\mu} \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \frac{\partial \psi(x)}{\partial x^\mu} \right) + |F(x)|^2 \\
&\quad - \frac{1}{2} (i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) - i \partial_\mu \phi^\dagger(x) V^\mu \phi(x)) + \frac{1}{2} \phi^\dagger(x) D \phi(x) \\
&\quad - \frac{1}{2} \bar{\psi}(x) \bar{\sigma}^\mu V_\mu \psi(x) + \frac{1}{\sqrt{2}} (i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x)) + \frac{1}{2} \frac{1}{2} \phi^\dagger(x) V^\mu V_\mu \phi(x) \\
&= \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left( -\bar{\psi}(x) \left( \frac{\bar{\partial}}{\partial x^\mu} - \frac{1}{2} i V_\mu \right) \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \left( \frac{\partial}{\partial x^\mu} + \frac{1}{2} i V_\mu \right) \psi(x) \right) + |F(x)|^2 \\
&\quad - \frac{1}{2} (i \phi^\dagger(x) V^\mu \partial_\mu \phi(x) - i \partial_\mu \phi^\dagger(x) V^\mu \phi(x)) + \frac{1}{2} \phi^\dagger(x) D \phi(x) \\
&\quad + \frac{1}{\sqrt{2}} (i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x)) + \frac{1}{2} \frac{1}{2} \phi^\dagger(x) V^\mu V_\mu \phi(x) \\
&= \left| \mathcal{D}_\mu \phi(x) \right|^2 + \frac{1}{2} i \left( -\bar{\psi}(x) \bar{\mathcal{D}}_\mu \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \mathcal{D}_\mu \psi(x) \right) + \frac{1}{\sqrt{2}} (i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x)) \\
&\quad + \frac{1}{2} \phi^\dagger(x) D \phi(x)
\end{aligned}$$

ここで、 $V = 2g \sum_{a=1}^N V^a t^a$  に注意して、

$$\begin{aligned}
& \left| \mathcal{D}_\mu \phi(x) \right|^2 + \frac{1}{2} i \left( -\bar{\psi}(x) \bar{\mathcal{D}}_\mu \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \mathcal{D}_\mu \psi(x) \right) + \frac{1}{\sqrt{2}} \left( i \bar{\psi}(x) \bar{\lambda} \phi(x) - i \phi^\dagger(x) \lambda \psi(x) \right) \\
& + \frac{1}{2} \phi^\dagger(x) D \phi(x) \\
& = \left| \mathcal{D}_\mu \phi(x) \right|^2 + \frac{1}{2} i \left( -\bar{\psi}(x) \bar{\mathcal{D}}_\mu \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \mathcal{D}_\mu \psi(x) \right) + \sqrt{2} g \left( i \bar{\psi}(x) \bar{\lambda}^a t^a \phi(x) - i \phi^\dagger(x) \lambda^a t^a \psi(x) \right) \\
& + g \phi^\dagger(x) D^a t^a \phi(x)
\end{aligned}$$

になる。故に、求めるゲージ相互作用は

$$\begin{aligned}
L &= \int \mathcal{L}(x) d^4x = \int \Phi^\dagger(x) \exp(V) \Phi(x) d^4x d^4\theta \\
\mathcal{L}(x) &= \left| \mathcal{D}_\mu \phi(x) \right|^2 + \frac{1}{2} i \left( -\bar{\psi}(x) \bar{\mathcal{D}}_\mu \bar{\sigma}^\mu \psi(x) + \bar{\psi}(x) \bar{\sigma}^\mu \mathcal{D}_\mu \psi(x) \right) \\
& + \sqrt{2} i g \left( \bar{\psi}(x) \bar{\lambda}^a t^a \phi(x) - \phi^\dagger(x) \lambda^a t^a \psi(x) \right) + g \phi^\dagger(x) D^a t^a \phi(x) \\
\text{with } \mathcal{D}_\mu &= \partial_\mu + \frac{1}{2} i V_\mu = \partial_\mu + i g \sum_{a=1}^N V_\mu^a t^a
\end{aligned}$$

になる。