

第7章 超ラグランジアン

超運動項

運動項(kinetic term)を超場を用いて表す。超ベクトル場の超対称性変換によると

$$\delta C(x) = -i(\varepsilon\chi(x) - \bar{\varepsilon}\bar{\chi}(x))$$

$$\delta\chi_\alpha(x) = -\left(2\varepsilon_\alpha m(x) - i(\sigma^\mu\bar{\varepsilon})_\alpha\left(V_\mu - i\frac{\partial C(x)}{\partial x^\mu}\right)\right), \quad \delta\bar{\chi}_\alpha(x) = -\left(2\bar{\varepsilon}_\alpha m^\dagger(x) + i(\varepsilon\sigma^\mu)_\alpha\left(V_\mu + i\frac{\partial C(x)}{\partial x^\mu}\right)\right)$$

$$\delta m(x) = -\bar{\varepsilon}\bar{\lambda}(x) - i\frac{\partial\chi(x)}{\partial x^\mu}\sigma^\mu\bar{\varepsilon}, \quad \delta m^\dagger(x) = -\varepsilon\lambda(x) + i\varepsilon\sigma^\mu\frac{\partial\bar{\chi}(x)}{\partial x^\mu}$$

$$\delta V_\mu = -i(\varepsilon\sigma^\mu\bar{\lambda}(x) - \lambda(x)\sigma^\mu\bar{\varepsilon}) - \frac{\partial}{\partial x_\mu}(\varepsilon\chi(x) + \bar{\varepsilon}\bar{\chi}(x))$$

$$\delta\lambda_\alpha(x) = -\frac{1}{2}i(\sigma^{\mu\nu}\varepsilon)_\alpha V_{\mu\nu} - i\varepsilon_\alpha D(x), \quad \delta\bar{\lambda}_\alpha(x) = -\frac{1}{2}i(\bar{\varepsilon}\sigma^{\mu\nu})_\alpha V_{\mu\nu} + i\bar{\varepsilon}_\alpha D(x)$$

$$\delta D(x) = -\frac{\partial}{\partial x^\nu}(\lambda(x)\sigma^\nu\bar{\varepsilon} + \varepsilon\sigma^\nu\bar{\lambda}(x))$$

である。ここに、一般的な成分表示

$$\begin{aligned} V(x, \theta, \bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + i\theta\theta m(x) - i\bar{\theta}\bar{\theta}m^\dagger(x) + \theta\sigma^\mu\bar{\theta}V_\mu \\ &\quad + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) - \frac{1}{2}i\bar{\sigma}^\mu\frac{\partial\chi(x)}{\partial x^\mu}\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) - \frac{1}{2}i\sigma^\mu\frac{\partial\bar{\chi}(x)}{\partial x^\mu}\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) - \frac{1}{2}\frac{\partial^2 C(x)}{\partial x^\mu\partial x_\mu}\right) \end{aligned}$$

である。

$$\delta D(x) = -\frac{\partial}{\partial x^\nu}(\lambda(x)\sigma^\nu\bar{\varepsilon} + \varepsilon\sigma^\nu\bar{\lambda}(x))$$

に注目すると、

$$\delta\int Dd^4x = -\int\frac{\partial}{\partial x^\nu}(\lambda(x)\sigma^\nu\bar{\varepsilon} + \varepsilon\sigma^\nu\bar{\lambda}(x))d^4x = -\int d\Sigma_\nu(\lambda(x)\sigma^\nu\bar{\varepsilon} + \varepsilon\sigma^\nu\bar{\lambda}(x))$$

$$\stackrel{t=\text{一定}}{=} -\int d^3x(\lambda(x)\sigma^0\bar{\varepsilon} + \varepsilon\sigma^0\bar{\lambda}(x))\Big|_{t=-\infty}^{t=\infty} = 0$$

$$(*)d\Sigma_\mu = \sum_{\nu\rho\tau} \frac{1}{3!}\varepsilon_{\mu\nu\rho\tau}dx^\nu \otimes dx^\rho \otimes dx^\tau = (dx^1 dx^2 dx^3, dx^2 dx^3 dx^0, dx^3 dx^0 dx^1, dx^0 dx^1 dx^2)$$

になるので、ベクトル超場の D-項は4次元積分すると超対称変換のもと不変になる。つまり、Lagrangian としての候補になる。

$$V(x, \theta, \bar{\theta})\Big|_D = V\text{の}\theta\theta\bar{\theta}\bar{\theta}\text{の係数}$$

とすると、

$$L \supset \int V(x, \theta, \bar{\theta})\Big|_D d^4x$$

と表せる。ここで、グラスマン数の積分を定義する：

$$\int d\theta^\alpha = \int d\bar{\theta}^\alpha = 0, \quad \int d\theta^\alpha \theta^\alpha = \int d\bar{\theta}^\alpha \bar{\theta}^\alpha = 1 \quad (\text{with no summation over } \alpha)$$

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta_\alpha \text{ so that } \int d^2\theta\theta\theta = 1, \quad d^2\bar{\theta} = -\frac{1}{4}d\bar{\theta}_\alpha d\bar{\theta}^\alpha \text{ so that } \int d^2\bar{\theta}\bar{\theta}\bar{\theta} = 1$$

$$d^4\theta = d^2\theta d^2\bar{\theta}$$

これを用いると、

$$\boxed{\begin{aligned} V(x, \theta, \bar{\theta}) \Big|_D &= V \text{ の } \theta\theta \text{ の係数} = \int d^4\theta V(x, \theta, \bar{\theta}) \\ L &\supset \int V(x, \theta, \bar{\theta}) \Big|_D d^4x = \int V(x, \theta, \bar{\theta}) d^4x d^4\theta \end{aligned}}$$

になる。

そこで、2つのカイラル超場を使用すれば

$$V(x, \theta, \bar{\theta}) = \Phi^\dagger(y, \theta)\Phi(y, \theta) \quad \text{with } \Phi(y, \theta) = \phi(y) - \sqrt{2}\theta\psi(y) + F(y)\theta\theta \quad \text{with } y = x - i\theta\sigma\bar{\theta}$$

である。

$$\Phi^\dagger(y, \theta)\Phi(y, \theta) = \Phi^\dagger(x - i\theta\sigma\bar{\theta}, \theta)\Phi(x - i\theta\sigma\bar{\theta}, \theta)$$

$$\Leftarrow \Phi(x - i\theta\sigma\bar{\theta}, \theta) = \Phi(x, \theta) + (-i\theta\sigma^\mu\bar{\theta})\frac{\partial\Phi(x, \theta)}{\partial x^\mu} + \frac{1}{2}(-i\theta\sigma^\nu\bar{\theta})(-i\theta\sigma^\mu\bar{\theta})\frac{\partial^2\Phi(x, \theta)}{\partial x^\mu\partial x^\nu}$$

$$= (\dots)^\dagger \cdot \left[\Phi(x, \theta) - i\theta\sigma^\mu\bar{\theta}\frac{\partial\Phi(x, \theta)}{\partial x^\mu} - \frac{1}{2}(\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta})\frac{\partial^2\Phi(x, \theta)}{\partial x^\mu\partial x^\nu} \right]$$

$$\downarrow \Leftarrow \Phi(x, \theta) = \phi(x) - \sqrt{2}\theta\psi(x) + F(x)\theta\theta$$

$$= (\dots)^\dagger \cdot \left[\phi(x) - \sqrt{2}\theta\psi(x) + F(x)\theta\theta - i\theta\sigma^\mu\bar{\theta}\frac{\partial(\phi(x) - \sqrt{2}\theta\psi(x))}{\partial x^\mu} - \frac{1}{2}(\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta})\frac{\partial^2\phi(x)}{\partial x^\mu\partial x^\nu} \right]$$

$$= -\frac{1}{2}(\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta})\left(\phi^\dagger(x)\frac{\partial^2\phi(x)}{\partial x^\mu\partial x^\nu} + \frac{\partial^2\phi^\dagger(x, \theta)}{\partial x^\mu\partial x_\mu}\phi(x) \right) + i\theta\sigma^\mu\bar{\theta}\frac{\partial\phi^\dagger(x)}{\partial x^\mu}(-i\theta\sigma^\nu\bar{\theta})\frac{\partial\phi(x)}{\partial x^\nu}$$

$$- \sqrt{2}\bar{\theta}\bar{\psi}(x)(-i\theta\sigma^\mu\bar{\theta})\frac{\partial(-\sqrt{2}\theta\psi(x))}{\partial x^\mu} + (i\theta\sigma^\mu\bar{\theta})\frac{\partial(-\sqrt{2}\bar{\theta}\bar{\psi}(x))}{\partial x^\mu}(-\sqrt{2}\theta\psi(x))$$

$$+ F^\dagger(x)\bar{\theta}\bar{\theta}F(x)\theta\theta + (\dots)$$

$$\Leftarrow (\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}$$

$$\Leftarrow \bar{\theta}\bar{\psi}(x)\theta\sigma^\mu\bar{\theta} \cdot \theta\psi(x) = \bar{\theta}_\alpha\bar{\psi}^\alpha(x)(\theta\sigma^\mu)_\beta\bar{\theta}^\beta \cdot \theta\psi(x) = -\frac{1}{2}\bar{\theta}\bar{\theta} \cdot \theta\sigma^\mu\bar{\psi}(x) \cdot \theta\psi(x)$$

$$= -\frac{1}{2}\bar{\theta}\bar{\theta} \cdot \theta^\alpha(\sigma^\mu\bar{\psi}(x))_\alpha \cdot (-\theta_\beta\psi^\beta(x)) = \frac{1}{4}\psi(x)\sigma^\mu\bar{\psi}(x)\theta\theta\bar{\theta}\bar{\theta}$$

$$= \frac{1}{2} \frac{1}{2} g^{\mu\nu} \theta\theta\bar{\theta}\bar{\theta} \left(-\phi^\dagger(x)\frac{\partial^2\phi(x)}{\partial x^\mu\partial x_\mu} - \frac{\partial^2\phi^\dagger(x, \theta)}{\partial x^\mu\partial x_\mu}\phi(x) + 2\frac{\partial\phi^\dagger(x)}{\partial x^\mu}\frac{\partial\phi(x)}{\partial x_\mu} \right)$$

$$- 2i \left(\frac{1}{4}\frac{\partial\psi(x)}{\partial x^\mu}\sigma^\mu\bar{\psi}(x) - \frac{1}{4}\psi(x)\sigma^\mu\frac{\partial\bar{\psi}(x)}{\partial x^\mu} \right) \theta\theta\bar{\theta}\bar{\theta} + |F(x)|^2\theta\theta\bar{\theta}\bar{\theta} + (\dots)$$

$$= \frac{1}{4} \theta \theta \theta \theta \left(-\phi^\dagger(x) \frac{\partial^2 \phi(x)}{\partial x^\mu \partial x_\mu} - \frac{\partial^2 \phi^\dagger(x, \theta)}{\partial x^\mu \partial x_\mu} \phi(x) + 2 \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 \right) +$$

$$+ \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) \theta \theta \theta \theta + |F(x)|^2 \theta \theta \theta \theta + (\dots)$$

以上から、

$$\Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D = \Phi^\dagger \Phi \text{ の } \theta \theta \theta \theta \text{ の係数}$$

は、

$$\begin{aligned} \Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D &= \Phi^\dagger \Phi \text{ の } \theta \theta \theta \theta \text{ の係数} \\ &= \frac{1}{4} \left(-\phi^\dagger(x) \frac{\partial^2 \phi(x)}{\partial x^\mu \partial x_\mu} - \frac{\partial^2 \phi^\dagger(x, \theta)}{\partial x^\mu \partial x_\mu} \phi(x) + 2 \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 \right) + \\ &\quad + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \\ &= \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 - \frac{1}{4} \frac{\partial}{\partial x^\mu} \left(\phi^\dagger(x) \frac{\partial \phi(x)}{\partial x_\mu} + \frac{\partial \phi^\dagger(x, \theta)}{\partial x_\mu} \phi(x) \right) + \\ &\quad + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \\ &= \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \\ &\quad - \frac{1}{4} \frac{\partial}{\partial x^\mu} \left(\phi^\dagger(x) \frac{\partial \phi(x)}{\partial x_\mu} + \frac{\partial \phi^\dagger(x, \theta)}{\partial x_\mu} \phi(x) \right) \end{aligned}$$

$$\begin{aligned} L &\supset \int \Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D d^4 x = \int \Phi^\dagger(y, \theta) \Phi(y, \theta) d^4 x d^4 \theta \\ &= \int \left[\left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \right. \\ &\quad \left. - \frac{1}{4} \frac{\partial}{\partial x^\mu} \left(\phi^\dagger(x) \frac{\partial \phi(x)}{\partial x_\mu} + \frac{\partial \phi^\dagger(x, \theta)}{\partial x_\mu} \phi(x) \right) \right] d^4 x \\ &\stackrel{\text{表面項無視}}{=} \int \left[\left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \right] d^4 x \end{aligned}$$

なので、最終的に、

$$L \supset \int \mathcal{L}(x) d^4 x \quad \text{with} \quad \mathcal{L}(x) = \Phi^\dagger(y, \theta) \Phi(y, \theta) \Big|_D = \int \Phi^\dagger(y, \theta) \Phi(y, \theta) d^4 \theta$$

$$\mathcal{L}(x) = \left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2$$

により、それぞれの場の運動項が求まった。

超ポテンシャル

カイラル超場の超対称性変換によると

$$\delta\phi = \sqrt{2}\varepsilon\psi, \quad \delta F = \sqrt{2}i\partial_\mu\psi\sigma^\mu\bar{\varepsilon}, \quad \delta\psi_\alpha = -\sqrt{2}i(\sigma^\mu\bar{\varepsilon})_\alpha \partial_\mu\phi + \sqrt{2}\varepsilon_\alpha F$$

$$\Phi(y, \theta) = \phi(y) - \sqrt{2}\theta\psi(y) + F(y)\theta\theta \quad \text{with } y = x - i\theta\sigma\bar{\theta}$$

である。さて、

$$\delta F = \sqrt{2}i\partial_\mu\psi\sigma^\mu\bar{\varepsilon} = \partial_\mu(\sqrt{2}i\psi\sigma^\mu\bar{\varepsilon})$$

に注目すると、

$$\delta \int F d^4x = \delta \int \partial_\mu(\sqrt{2}i\psi\sigma^\mu\bar{\varepsilon}) d^4x = \int d\Sigma_\mu(\sqrt{2}i\psi\sigma^\mu\bar{\varepsilon}) \Big|_{t=-\infty}^{t=\infty} = \int d^3x(\sqrt{2}i\psi\sigma^0\bar{\varepsilon}) \Big|_{t=-\infty}^{t=\infty} = 0$$

$$(*) d\Sigma_\mu = \sum_{\nu\rho\tau} \frac{1}{3!} \varepsilon_{\mu\nu\rho\tau} dx^\nu \otimes dx^\rho \otimes dx^\tau = (dx^1 dx^2 dx^3, dx^2 dx^3 dx^0, dx^3 dx^0 dx^1, dx^0 dx^1 dx^2)$$

になるので、カイラル超場の F-項は4次元積分すると超対称変換のもと不変になる。つまり、Lagrangian としての候補になる。

$$W(y, \theta) \Big|_F \quad (\text{for } y = x - i\theta\sigma\bar{\theta}) = W \text{ の } \theta\theta \text{ の係数} = W(x, \theta) \Big|_F$$

とすると、

$$L \supset \int W(y, \theta) \Big|_F d^4x = \int W(x, \theta) \Big|_F d^4x$$

と表せる。この W を**超ポテンシャル**という。ここで、グラスマン数の積分を用いると、

$$\begin{aligned} W(y, \theta) \Big|_F &= W \text{ の } \theta\theta \text{ の係数} = \int d^2\theta W(x, \theta) \\ L \supset \int W(y, \theta) \Big|_F d^4x &= \int W(x, \theta) d^4x d^2\theta \end{aligned}$$

になる。

超ポテンシャルを成分表示するには、

$$\begin{aligned} W(\Phi) &= W(\phi(y) - \sqrt{2}\theta\psi(y) + F(y)\theta\theta) \\ &= W(\phi) + (-\sqrt{2}\theta\psi_i(y) + F_i(y)\theta\theta) \frac{\partial W(\phi)}{\partial \phi_i} + \frac{1}{2!} (-\sqrt{2}\theta\psi_i(y) + F_i(y)\theta\theta) \\ &\quad \cdot (-\sqrt{2}\theta\psi_j(y) + F_j(y)\theta\theta) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} \\ &= W(\phi) + (-\sqrt{2}\theta\psi_i(y) + F_i(y)\theta\theta) \frac{\partial W(\phi)}{\partial \phi_i} + \frac{1}{2} (-\sqrt{2}\theta\psi_i(y)) (-\sqrt{2}\theta\psi_j(y)) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} \\ W(\Phi) &= F_i(y)\theta\theta \frac{\partial W(\phi)}{\partial \phi_i} + \theta\psi_i(y)\theta\psi_j(y) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} + \dots \Leftarrow \theta\psi_i\theta\psi_j = -\frac{1}{2}(\theta\theta)(\psi_i\psi_j) \\ &= F_i(y)\theta\theta \frac{\partial W(\phi)}{\partial \phi_i} - \frac{1}{2}(\psi_i\psi_j)\theta\theta \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} + \dots \end{aligned}$$

$$W(\Phi) = \left[F_i(y) \frac{\partial W(\phi)}{\partial \phi_i} - \frac{1}{2} \psi_i(y) \psi_j(y) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} \right] \theta \theta + \dots$$

$$W(\Phi)|_{\theta\theta} = F_i(y) \frac{\partial W(\phi)}{\partial \phi_i} - \frac{1}{2} \psi_i(y) \psi_j(y) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j}$$

になる。故に、

$$L \supset \int \mathcal{L}(x) d^4x \quad \text{with} \quad \mathcal{L}(x) = W(y, \theta)|_F = \int W(x, \theta) d^2\theta$$

$$\mathcal{L}(x) = F_i(x) \frac{\partial W(\phi)}{\partial \phi_i} - \frac{1}{2} \psi_i(x) \psi_j(x) \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j}$$

がわかる。

超ラグランジアン

運動項の

$$L \supset \int \Phi^\dagger(y, \theta) \Phi(y, \theta)|_D d^4x = \int \Phi^\dagger(y, \theta) \Phi(y, \theta) d^4x d^4\theta$$

$$= \int \left[\left| \frac{\partial \phi(x)}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi(x) \sigma^\mu \frac{\partial \bar{\psi}(x)}{\partial x^\mu} - \frac{\partial \psi(x)}{\partial x^\mu} \sigma^\mu \bar{\psi}(x) \right) + |F(x)|^2 \right] d^4x$$

とあわせると、

$$\begin{aligned} \mathcal{L} &= \Phi^\dagger(y, \theta) \Phi(y, \theta)|_D + (W(y, \theta)|_F + \text{h.c.}) \\ &= \int \Phi^\dagger(y, \theta) \Phi(y, \theta) d^4\theta + \left(\int W(x, \theta) d^2\theta + \text{h.c.} \right) \\ &= \left| \frac{\partial \phi_i}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi_i \sigma^\mu \frac{\partial \bar{\psi}_i}{\partial x^\mu} - \frac{\partial \psi_i}{\partial x^\mu} \sigma^\mu \bar{\psi}_i \right) + |F_i|^2 + \left(F_i \frac{\partial W(\phi)}{\partial \phi_i} - \frac{1}{2} \psi_i \psi_j \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} + \text{h.c.} \right) \end{aligned}$$

と表すことができる。ここで、 F_i の微分項が無いので、ラグランジュ方程式：

$$\frac{\partial \mathcal{L}}{\partial F_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu F_i)} = 0 \Rightarrow F_i^\dagger + \frac{\partial W(\phi)}{\partial \phi_i} = 0 \Rightarrow F_i^\dagger = -\frac{\partial W(\phi)}{\partial \phi_i}$$

F_i^\dagger についても同様なのでこれを元の \mathcal{L} に代入して、

$$\mathcal{L} = \left| \frac{\partial \phi_i}{\partial x^\mu} \right|^2 + \frac{1}{2} i \left(\psi_i \sigma^\mu \frac{\partial \bar{\psi}_i}{\partial x^\mu} - \frac{\partial \psi_i}{\partial x^\mu} \sigma^\mu \bar{\psi}_i \right) - \left| \frac{\partial W(\phi)}{\partial \phi_i} \right|^2 - \frac{1}{2} \left(\psi_i \psi_j \frac{\partial W(\phi)}{\partial \phi_i \partial \phi_j} + \text{h.c.} \right)$$

を得る。特に、

$$W(\phi) = \frac{1}{2} \mu_{ij} \phi_i \phi_j + \frac{1}{3} f_{ijk} \phi_i \phi_j \phi_k \quad (\Rightarrow \mu_{ij} = \mu_{ji}, f_{ijk} = f_{jik} = \dots)$$

とするとき、

$$\begin{aligned}
\frac{\partial W(\phi)}{\partial \phi_i} &= \frac{1}{2}(\delta_{ik}\mu_{k\ell}\phi_\ell + \delta_{i\ell}\phi_k\mu_{k\ell}) + \frac{1}{3}(f_{k\ell m}\delta_{ik}\phi_\ell\phi_m + f_{k\ell m}\delta_{i\ell}\phi_k\phi_m + f_{k\ell m}\delta_{im}\phi_k\phi_\ell) \\
&= \frac{1}{2}(\mu_{i\ell}\phi_\ell + \phi_k\mu_{ki}) + \frac{1}{3}(f_{i\ell m}\phi_\ell\phi_m + f_{kim}\phi_k\phi_m + f_{k\ell i}\phi_k\phi_\ell) \\
&= \mu_{i\ell}\phi_\ell + f_{i\ell m}\phi_\ell\phi_m \\
\frac{\partial^2 W(\phi)}{\partial \phi_i \partial \phi_j} &= \frac{\partial}{\partial \phi_j}(\mu_{i\ell}\phi_\ell + f_{i\ell m}\phi_\ell\phi_m) = \mu_{ij} + 2f_{ijm}\phi_m
\end{aligned}$$

なので、

$$\mathcal{L} = \left| \frac{\partial \phi_i}{\partial x^\mu} \right|^2 + \frac{1}{2}i \left(\psi_i \sigma^\mu \frac{\partial \bar{\psi}_i}{\partial x^\mu} - \frac{\partial \psi_i}{\partial x^\mu} \sigma^\mu \bar{\psi}_i \right) - |\mu_{i\ell}\phi_\ell + f_{i\ell m}\phi_\ell\phi_m|^2 - \frac{1}{2}((\mu_{ij} + 2f_{ijm}\phi_m)\psi_i\psi_j + \text{h.c.})$$

質量項は、

$$\begin{aligned}
m_{Bij}^2 &\equiv -\frac{\partial^2 \mathcal{L}}{\partial \phi_i^\dagger \partial \phi_j} = -\frac{\partial^2 (-\mu_{k\ell}\phi_\ell^\dagger \mu_{k\ell'}\phi_{\ell'})}{\partial \phi_i^\dagger \partial \phi_j} = \mu_{ki}\mu_{kj} \stackrel{\mu_{ik}=\mu_i\delta_{ik}}{\Rightarrow} m_{ii}^2 = \mu_i^2 \\
m_{Fij} &\equiv -\frac{\partial^2 \mathcal{L}}{\partial \psi_i \partial \psi_j} = -\frac{\partial^2 \left(-\frac{1}{2}\mu_{k\ell}\psi_k\psi_\ell \right)}{\partial \psi_i \partial \psi_j} = \mu_{ij} \stackrel{\mu_{ij}=\mu_i\delta_{ij}}{\Rightarrow} m_{ii} = \mu_i
\end{aligned}$$

となり、 $\mu_{ij} = \mu_i\delta_{ij}$ のときには、同じ質量を持つことがわかる。また、

$$\sum_{ik} \mu_{ki}\mu_{ki} = \sum_i m_{Bii}^2 = \sum_{ij} m_{Fij}^2 \Rightarrow \text{Tr}(m_B^2) = \text{Tr}(m_F^T m_F)$$

が成立する。