

第6章 超ゲージ対称性

Non-abelian ゲージ群

「第1章 場の変換性と演算子」-「変数 X が同じとき」より、 T^a を generators にもつ Non-abelian 群の下で、

$$\begin{aligned} U^\dagger \hat{\phi}(x) U &= \Omega \hat{\phi}(x) \quad \text{with } U = \exp(iT(a)) \quad \& \quad \Omega = \exp(it(a)) \\ &\Leftarrow \exp(-ig\kappa^a T^a) \hat{\phi}(x) \exp(ig\kappa^a T^a) = \exp(ig\kappa^a t^a) \hat{\phi}(x) \\ \delta \hat{\phi}(X) &= [ig\kappa^a T^a, \hat{\phi}(X)] = -ig\kappa^a t^a \hat{\phi}(x) \end{aligned}$$

に注意して、カイラル超場 Φ が

$$\begin{aligned} \Phi'(x) &= \exp(2ig\Lambda^a T^a) \Phi(x) \exp(-2ig\Lambda^a T^a) = \exp(-2ig\Lambda^a t^a) \Phi(x) \\ \text{with } [T^a, T^b] &= if^{abc} T^c \quad \& \quad [t^a, t^b] = if^{abc} t^c \end{aligned}$$

で変換するとき、群の変換 U で不変になる超運動項は、ベクトルゲージ超場 V^a を導入すれば

$$\Phi^\dagger(x) \exp(2gV^a t^a) \Phi(x)$$

である。ここに、 V^a は不変性の要請により、

$$\begin{aligned} \Phi^\dagger(x) \exp(2gV'^a t^a) \Phi'(x) &= \left(\exp(-2ig\Lambda^a t^a) \Phi(x) \right)^\dagger \exp(2gV'^a t^a) \exp(-2ig\Lambda^a t^a) \Phi(x) \\ &= \Phi^\dagger(x) \exp(2ig\Lambda^a t^a) \exp(2gV'^a t^a) \exp(-2ig\Lambda^a t^a) \Phi(x) \Leftrightarrow \Phi^\dagger(x) \exp(2gV^a t^a) \Phi(x) \end{aligned}$$

より

$$\exp(2gV'^a t^a) = \exp(-2ig\Lambda^a t^a) \exp(2gV^a t^a) \exp(2ig\Lambda^a t^a)$$

である。以降、省略形を用いる。

$$\exp(V') = \exp(-i\Lambda^\dagger) \exp(V) \exp(i\Lambda) \quad \text{with } V = 2g \sum_{a=1}^N V^a t^a \quad \& \quad \Lambda = 2g \sum_{a=1}^N \Lambda^a t^a$$

さて、通常、微小変換 $\delta V: V' = V + \delta V$ と Λ を考えると、

$$\begin{aligned} \exp(V') &= \exp(V + \delta V) = 1 + V + \delta V + \frac{1}{2!}(V^2 + \delta V V + V \delta V) + \frac{1}{3!}(V^3 + \delta V V^2 + V \delta V V + V^2 \delta V) + \dots \\ \exp(V') &= \exp(-i\Lambda^\dagger) \exp(V) \exp(i\Lambda) = 1 + V + \frac{1}{2!}V^2 + \frac{1}{3!}V^3 + i(\Lambda - \Lambda^\dagger) + i(V\Lambda - \Lambda^\dagger V) \\ &\quad + \frac{i}{2!}(V^2\Lambda - \Lambda^\dagger V^2) + \frac{i}{3!}(V^3\Lambda - \Lambda^\dagger V^3) + \dots \end{aligned}$$

の展開を用いる。これから、摂動的な解

$$\delta V = i(\Lambda - \Lambda^\dagger) + (a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) + (a_2\Lambda + b_2\Lambda^\dagger)V^2 + V^2(c_2\Lambda + d_2\Lambda^\dagger) + V(e_2\Lambda + f_2\Lambda^\dagger)V$$

を用いて、

$$\delta V + \frac{1}{2!}(\delta V V + V \delta V) + \frac{1}{3!}(\delta V V^2 + V \delta V V + V^2 \delta V) = i(\Lambda - \Lambda^\dagger) + i(V\Lambda - \Lambda^\dagger V) + \frac{i}{2!}(V^2\Lambda - \Lambda^\dagger V^2)$$

を V^2 までで解くことになる。つまり、

$$\begin{aligned}
& i(\Lambda - \Lambda^\dagger) + (a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) + (a_2\Lambda + b_2\Lambda^\dagger)V^2 + V^2(c_2\Lambda + d_2\Lambda^\dagger) + V(e_2\Lambda + f_2\Lambda^\dagger)V \\
& + \frac{1}{2!} \left(i(\Lambda - \Lambda^\dagger) + (a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) \right) V + \frac{1}{2!} V \left(i(\Lambda - \Lambda^\dagger) + (a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) \right) \\
& + \frac{1}{3!} i(\Lambda - \Lambda^\dagger)V^2 + \frac{1}{3!} V^2 i(\Lambda - \Lambda^\dagger) + \frac{1}{3!} Vi(\Lambda - \Lambda^\dagger)V \\
& \stackrel{\text{要請}}{=} i(\Lambda - \Lambda^\dagger) + i(V\Lambda - \Lambda^\dagger V) + \frac{i}{2!} (V^2\Lambda - \Lambda^\dagger V^2) + \frac{i}{3!} (V^3\Lambda - \Lambda^\dagger V^3)
\end{aligned}$$

V の一次の項から

$$\begin{aligned}
& (a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) + \frac{1}{2!} \left(i(\Lambda - \Lambda^\dagger) \right) V + \frac{1}{2!} V \left(i(\Lambda - \Lambda^\dagger) \right) = i(V\Lambda - \Lambda^\dagger V) \\
& \Rightarrow a_1 + \frac{i}{2} = 0, d_1 - \frac{i}{2} = 0, b_1 - \frac{i}{2} = -i, c_1 + \frac{i}{2} = i \Rightarrow a_1 = -\frac{i}{2}, d_1 = \frac{i}{2}, b_1 = -\frac{i}{2}, c_1 = \frac{i}{2} \\
& \Rightarrow \delta^{(1)}V = i(\Lambda - \Lambda^\dagger) - \frac{i}{2}(\Lambda + \Lambda^\dagger)V + \frac{i}{2}V(\Lambda + \Lambda^\dagger) = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger]
\end{aligned}$$

V の二次の項から

$$\begin{aligned}
& \frac{1}{2!} \left((a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) \right) V + \frac{1}{2!} V \left((a_1\Lambda + b_1\Lambda^\dagger)V + V(c_1\Lambda + d_1\Lambda^\dagger) \right) \\
& + \frac{1}{3!} i(\Lambda - \Lambda^\dagger)V^2 + \frac{1}{3!} V^2 i(\Lambda - \Lambda^\dagger) + \frac{1}{3!} Vi(\Lambda - \Lambda^\dagger)V = \frac{i}{2!} (V^2\Lambda - \Lambda^\dagger V^2) \\
& \Rightarrow a_2 + \frac{1}{2!}a_1 + \frac{1}{3!}i = 0, b_2 + \frac{1}{2!}b_1 - \frac{1}{3!}i = -\frac{i}{2!}, c_2 + \frac{1}{2!}c_1 + \frac{1}{3!}i = \frac{i}{2!}, d_2 + \frac{1}{2!}d_1 - \frac{1}{3!}i = 0 \\
& e_2 + \frac{1}{2!}c_1 + \frac{1}{2!}a_1 + \frac{1}{3!}i = 0, f_2 + \frac{1}{2!}d_1 + \frac{1}{2!}b_1 - \frac{1}{3!}i = 0 \\
& \Rightarrow a_2 = -\frac{1}{2!}a_1 - \frac{i}{3!} = -\frac{1}{2!} \left(-\frac{i}{2} \right) - \frac{i}{3!} = \left(\frac{1}{4} - \frac{1}{6} \right) i = \frac{1}{12}i \\
& b_2 = -\frac{i}{2!} + \frac{i}{3!} - \frac{1}{2!}b_1 = -\frac{i}{2!} + \frac{i}{3!} - \frac{1}{2!} \left(-\frac{i}{2} \right) = \left(-\frac{1}{2} + \frac{1}{6} + \frac{1}{4} \right) i = -\frac{1}{12}i \\
& c_2 = \frac{i}{2!} - \frac{i}{3!} - \frac{1}{2!}c_1 = \frac{i}{2!} - \frac{i}{3!} - \frac{1}{2!} \frac{i}{2} = \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{4} \right) i = \frac{1}{12}i \\
& d_2 = \frac{1}{3!}i - \frac{1}{2!}d_1 = \frac{1}{3!}i - \frac{1}{2!} \frac{i}{2} = \left(\frac{1}{6} - \frac{1}{4} \right) i = -\frac{1}{12}i \\
& e_2 = -\frac{1}{2!}c_1 - \frac{1}{2!}a_1 - \frac{i}{3!} = -\frac{1}{2!} \frac{i}{2} - \frac{1}{2!} \left(-\frac{i}{2} \right) - \frac{i}{3!} = -\frac{i}{6} \\
& f_2 = -\frac{1}{2!}d_1 - \frac{1}{2!}b_1 + \frac{i}{3!} = -\frac{1}{2!} \frac{i}{2} - \frac{1}{2!} \left(-\frac{i}{2} \right) + \frac{i}{3!} = \frac{i}{6} \\
& \Rightarrow a_2 = \frac{1}{12}i, b_2 = -\frac{1}{12}i, c_2 = \frac{1}{12}i, d_2 = -\frac{1}{12}i, e_2 = -\frac{i}{6}, f_2 = \frac{i}{6}
\end{aligned}$$

これらより、

$$\begin{aligned}
\Rightarrow \delta^{(2)}V &= \frac{1}{12}i(\Lambda - \Lambda^\dagger)V^2 + \frac{1}{12}iV^2(\Lambda - \Lambda^\dagger) + \frac{i}{6}V(-\Lambda + \Lambda^\dagger)V \\
&= \frac{1}{12}i(\Lambda - \Lambda^\dagger)V^2 - \frac{i}{12}V(\Lambda - \Lambda^\dagger)V + \frac{1}{12}iV^2(\Lambda - \Lambda^\dagger) - \frac{i}{12}V(\Lambda - \Lambda^\dagger)V \\
&= \frac{1}{12}i[\Lambda - \Lambda^\dagger, V]V + \frac{1}{12}iV[V, \Lambda - \Lambda^\dagger] = \frac{1}{12}i[[\Lambda - \Lambda^\dagger, V], V]
\end{aligned}$$

以上から、

$$\delta V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \frac{1}{12}i[[\Lambda - \Lambda^\dagger, V], V] + (\dots)$$

Jacobi identity を用いれば、

$$\begin{aligned}
&[[\Lambda - \Lambda^\dagger, V], V] + [[V, V], \Lambda - \Lambda^\dagger] + [[V, \Lambda - \Lambda^\dagger], V] = 0 \\
\Rightarrow [[\Lambda - \Lambda^\dagger, V], V] &= [V, [V, \Lambda - \Lambda^\dagger]]
\end{aligned}$$

に注意する。

ここで、共変微分演算子を ∇_α であらわし、

$$(\nabla_\alpha \Phi(x))' = \exp(-i\Lambda)(\nabla_\alpha \Phi(x)) \text{ with } \Phi'(x) = \exp(-i\Lambda)\Phi(x)$$

とする。これから、

$$\nabla'_\alpha \Phi'(x) = \exp(-i\Lambda)(\nabla_\alpha \Phi(x)) \Rightarrow \nabla'_\alpha = \exp(-i\Lambda)\nabla_\alpha \exp(i\Lambda)$$

が要請される。この性質を満たすのは、

$$\nabla_\alpha \equiv \exp(-V)D_\alpha \exp(V)$$

である：

$$\begin{aligned}
\nabla'_\alpha &= \exp(-V')D_\alpha \exp(V') = \exp(-i\Lambda)\exp(-V)\exp(i\Lambda^\dagger)D_\alpha(\exp(-i\Lambda^\dagger)\exp(V)\exp(i\Lambda)) \\
&= \exp(-i\Lambda)\exp(-V)D_\alpha(\exp(V)\exp(i\Lambda)) \Leftarrow 0 = (\bar{D}_\alpha \Lambda)^\dagger = D_\alpha \Lambda^\dagger \\
&= \exp(-i\Lambda)(\exp(-V)D_\alpha \exp(V))\exp(i\Lambda) = \exp(-i\Lambda)\nabla_\alpha \exp(i\Lambda)
\end{aligned}$$

一方、通常共変微分は

$$\begin{aligned}
\psi(x)' &= U\psi(x) \\
G'_\mu(x) &= UG_\mu(x)U(x)^\dagger + iU(x)\partial_\mu U(x)^\dagger = UiD_\mu U(x)^\dagger \\
iD_\mu &= i\partial_\mu + G_\mu(x)
\end{aligned}$$

であるので、

$$\begin{aligned}
i\nabla'_\alpha &= Ui\nabla_\alpha U^\dagger = iU(\exp(-V)D_\alpha \exp(V))U^\dagger = Ui\mathbf{D}_\mu U^\dagger \text{ with } U = \exp(-i\Lambda) \\
i\mathbf{D}_\mu &\equiv i\partial_\alpha + \mathbf{G}_\alpha \equiv i\exp(-V)D_\alpha \exp(V) = iD_\alpha + i\exp(-V)(D_\alpha \exp(V)) \\
\Rightarrow \partial_\alpha &= D_\alpha \quad \& \quad \mathbf{G}_\alpha = i\exp(-V)(D_\alpha \exp(V)) = i\exp(-V)(\partial_\alpha \exp(V))
\end{aligned}$$

この対応より、

$$\mathbf{G}_\alpha = i \exp(-V)(D_\alpha \exp(V))$$

はゲージ接続に対応する。abelian の時のカイラルゲージ超場を利用すれば、

$$\begin{aligned} \mathbf{G}_\alpha &= i \exp(-V)(D_\alpha \exp(V)) \stackrel{\text{abelian}}{=} i D_\alpha V \stackrel{a=0, t^0=1}{=} i D_\alpha (2g V^0 t^0) = 2ig D_\alpha V^0 \\ W_\alpha &\stackrel{\text{abelian}}{=} \bar{D}_\beta \bar{D}^\beta D_\alpha V^0 \stackrel{\text{non-abelian}}{=} \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V)(D_\alpha \exp(V)) \end{aligned}$$

non-abelian の時に

$$\begin{aligned} W_\alpha(y) &= \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x))(D_\alpha \exp(V(x))) \\ \text{with } V(x) &= 2g \sum_{a=1}^N V^a(x) t^a, \quad y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} + i(\theta\sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu} \end{aligned}$$

と定義すればよい。

Abelian ゲージ固定

さて、ゲージ場は

$$\delta V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \frac{1}{12}i[[\Lambda - \Lambda^\dagger, V], V] + (\dots)$$

の変換性を受ける。abelian の場合は、

$$\delta V = i(\Lambda - \Lambda^\dagger)$$

であるが、成分表記をするため、

$$\begin{aligned} \Lambda(x - i\theta\sigma\bar{\theta}, \theta) &= \Lambda(x, \theta) + (-i\theta\sigma^\mu\bar{\theta}) \frac{\partial\Lambda(x, \theta)}{\partial x^\mu} + \frac{1}{2}(-i\theta\sigma^\nu\bar{\theta})(-i\theta\sigma^\mu\bar{\theta}) \frac{\partial^2\Lambda(x, \theta)}{\partial x^\mu\partial x^\nu} \\ &= \phi(x) - \sqrt{2}\theta\psi(x) + F\theta\theta - i\theta\sigma^\mu\bar{\theta} \frac{\partial(\phi(x) - \sqrt{2}\theta\psi(x))}{\partial x^\mu} - \frac{1}{2}(\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta}) \frac{\partial^2\phi(x, \theta)}{\partial x^\mu\partial x_\mu} \\ &= \phi - \sqrt{2}\theta\psi + F\theta\theta - i\theta\sigma^\mu\bar{\theta} \frac{\partial\phi}{\partial x^\mu} + i\sqrt{2} \frac{\partial(\theta\sigma^\mu\bar{\theta}\theta\psi)}{\partial x^\mu} - \frac{1}{2}(\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta}) \frac{\partial^2\phi(x)}{\partial x^\mu\partial x^\nu} \\ &\Leftrightarrow (\theta\sigma^\nu\bar{\theta})(\theta\sigma^\mu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta} \\ &\Leftrightarrow \theta\sigma^\mu\bar{\theta} \cdot \theta\psi = \psi^\beta\theta_\beta\theta^\alpha(\sigma^\mu\bar{\theta})_\alpha = -\frac{1}{2}\psi\sigma^\mu\bar{\theta}\theta\theta \\ &= \phi - \sqrt{2}\theta\psi + F\theta\theta - i\theta\sigma^\mu\bar{\theta} \frac{\partial\phi}{\partial x^\mu} + i\sqrt{2}\left(-\frac{1}{2}\right)\frac{\partial\psi}{\partial x^\mu}\sigma^\mu\bar{\theta}\theta\theta - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta} \frac{\partial^2\phi}{\partial x^\mu\partial x_\mu} \\ &= \phi - \sqrt{2}\theta\psi + F\theta\theta - i\theta\sigma^\mu\bar{\theta} \frac{\partial\phi}{\partial x^\mu} - i\frac{1}{\sqrt{2}}\frac{\partial\psi}{\partial x^\mu}\sigma^\mu\bar{\theta}\theta\theta - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta} \frac{\partial^2\phi}{\partial x^\mu\partial x_\mu} \end{aligned}$$

なので、

$$i(\Lambda - \Lambda^\dagger) = i\phi - i\sqrt{2}\theta\psi + iF\theta\theta + \theta\sigma^\mu\bar{\theta}\frac{\partial\phi}{\partial x^\mu} + \frac{1}{\sqrt{2}}\frac{\partial\psi}{\partial x^\mu}\sigma^\mu\bar{\theta}\theta\theta - \frac{1}{4}i\theta\theta\bar{\theta}\bar{\theta}\frac{\partial^2\phi}{\partial x^\mu\partial x_\mu} \\ - i\phi^\dagger + i\sqrt{2}\bar{\psi}\bar{\theta} - iF^\dagger\bar{\theta}\bar{\theta} + \theta\sigma^\mu\bar{\theta}\frac{\partial\phi^\dagger}{\partial x^\mu} + \frac{1}{\sqrt{2}}\theta\sigma^\mu\frac{\partial\bar{\psi}}{\partial x^\mu}\bar{\theta}\bar{\theta} + i\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\frac{\partial^2\phi^\dagger}{\partial x^\mu\partial x_\mu}$$

$$\delta V = \delta C + i\theta\delta\chi - i\bar{\theta}\delta\bar{\chi} + i\theta\theta\delta m - i\bar{\theta}\bar{\theta}\delta m^\dagger + \theta\sigma^\mu\bar{\theta}\delta V_\mu \\ + i\theta\bar{\theta}\bar{\theta}\left(\delta\bar{\lambda} - \frac{1}{2}i\bar{\sigma}^\mu\frac{\partial\delta\chi}{\partial x^\mu}\right) - i\bar{\theta}\theta\theta\left(\delta\lambda - \frac{1}{2}i\sigma^\mu\frac{\partial\delta\bar{\chi}}{\partial x^\mu}\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(\delta D - \frac{1}{2}\frac{\partial^2\delta C}{\partial x^\mu\partial x_\mu}\right)$$

から、

$$\delta C = i(\phi - \phi^\dagger), \delta\chi = -\sqrt{2}\psi, \delta m = F, \delta V_\mu = \frac{\partial}{\partial x^\mu}(\phi + \phi^\dagger), \delta\lambda = 0, \delta D = 0$$

がわかる。そのため、特別なゲージパラメータ (ϕ, ψ, F) として、

$$C' = C + i(\phi - \phi^\dagger) = 0, \chi' = \chi - \sqrt{2}\psi = 0, m' = m + F = 0$$

を選ぶことができる。その結果、与えられた (C, χ, m) に対して、常にこの成分を0にするよう

なゲージパラメータ (ϕ, ψ, F) を選ぶことができるので、変換後の V' は

$$V'(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}V'_\mu(x) + i\theta\theta\bar{\theta}\lambda'(x) - i\bar{\theta}\bar{\theta}\theta\lambda'(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D'(x)$$

と表すことができる、この V' を改めて V と表記すれば、

$$V_{WS}(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta}V_\mu(x) + i\theta\theta\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)$$

これを Wess-Zumino ゲージと呼ぶ。このとき、

$$V_{WS}(x, \theta, \bar{\theta})V_{WS}(x, \theta, \bar{\theta})V_{WS}(x, \theta, \bar{\theta}) = 0$$

である。

Non-abelian ゲージ固定

対応するゲージ変換

$$\delta V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \frac{1}{12}i[[\Lambda - \Lambda^\dagger, V], V] + (\dots)$$

に注意すると、non-abelian ゲージ固定は

$$V_{WS}V_{WS}V_{WS} = 0 \stackrel{\text{non-abelian}}{\Rightarrow} V^aV^bV^c = 0 \Rightarrow V^aV^b\Lambda^c = 0 \Rightarrow [[\Lambda - \Lambda^\dagger, V], V] + (\dots) = 0$$

の要請が必要となる。この結果、

$$\delta V_{WS} = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V_{WS}, \Lambda + \Lambda^\dagger]$$

になる。

Non-abelian ゲージ超場

共変微分演算子を ∇_α であらわし、

$$(\nabla_\alpha \Phi(x))' = \exp(-i\Lambda)(\nabla_\alpha \Phi(x)) \text{ with } \Phi'(x) = \exp(-i\Lambda)\Phi(x)$$

$$V = 2g \sum_{a=1}^N V^a t^a \quad \& \quad \Lambda = 2g \sum_{a=1}^N \Lambda^a t^a$$

とする。これから、

$$\nabla'_\alpha \Phi'(x) = \exp(-i\Lambda)(\nabla_\alpha \Phi(x)) \Rightarrow \nabla'_\alpha = \exp(-i\Lambda) \nabla_\alpha \exp(i\Lambda)$$

が要請される。この性質を満たすのは、

$$\nabla_\alpha \equiv \exp(-V) D_\alpha \exp(V)$$

である：

$$\begin{aligned} \nabla'_\alpha &= \exp(-V') D_\alpha \exp(V') = \exp(-i\Lambda) \exp(-V) \exp(i\Lambda^\dagger) D_\alpha (\exp(-i\Lambda^\dagger) \exp(V) \exp(i\Lambda)) \\ &= \exp(-i\Lambda) \exp(-V) D_\alpha (\exp(V) \exp(i\Lambda)) \Leftarrow 0 = (\bar{D}_\alpha \Lambda)^\dagger = D_\alpha \Lambda^\dagger \\ &= \exp(-i\Lambda) (\exp(-V) D_\alpha \exp(V)) \exp(i\Lambda) = \exp(-i\Lambda) \nabla_\alpha \exp(i\Lambda) \end{aligned}$$

一方、通常の変微分は

$$\begin{aligned} \psi(x)' &= U\psi(x) \\ G'_\mu(x) &= U G_\mu(x) U(x)^\dagger + iU(x) \partial_\mu U(x)^\dagger = U i D_\mu U(x)^\dagger \\ iD'_\mu &= i\partial_\mu + G'_\mu(x) \end{aligned}$$

であるので、

$$\begin{aligned} i\nabla'_\alpha &= U i \nabla_\alpha U^\dagger = iU (\exp(-V) D_\alpha \exp(V)) U^\dagger = U i \mathbf{D}_\alpha U^\dagger \text{ with } U = \exp(-i\Lambda) \\ i\mathbf{D}_\mu &\equiv i\partial_\mu + \mathbf{G}_\alpha \equiv i \exp(-V) D_\alpha \exp(V) = i D_\alpha + i \exp(-V) (D_\alpha \exp(V)) \\ &\Rightarrow \partial_\alpha = D_\alpha \quad \& \quad \mathbf{G}_\alpha = i \exp(-V) (D_\alpha \exp(V)) = i \exp(-V) (\partial_\alpha \exp(V)) \end{aligned}$$

この対応より、

$$\mathbf{G}_\alpha = i \exp(-V) (D_\alpha \exp(V))$$

はゲージ接続に対応する。abelian の時のカイラルゲージ超場を利用すれば、

$$\begin{aligned} \mathbf{G}_\alpha &= i \exp(-V) (D_\alpha \exp(V)) \stackrel{\text{abelian}}{=} i D_\alpha V \stackrel{\text{abelian}}{a=0, t^0=I} = i D_\alpha (2g V^0 t^0) = 2ig D_\alpha V^0 \\ W_\alpha &\stackrel{\text{abelian}}{=} \bar{D}_\beta \bar{D}^\beta D_\alpha V^0 \stackrel{\text{non-abelian}}{=} \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V) (D_\alpha \exp(V)) \end{aligned}$$

non-abelian の時に

$$W_\alpha(y) = \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x)) (D_\alpha \exp(V(x)))$$

$$\text{with } V(x) = 2g \sum_{a=1}^N V^a(x) t^a, \quad y^\mu = x^\mu - i\theta \sigma^\mu \bar{\theta}$$

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}$$

と定義すればよい。

さて、Abelian の場合には、

$$W_\alpha(y) \equiv \bar{D}_\beta \bar{D}^\beta D_\alpha V(x) = 4i\lambda_\alpha(y) - \left[4D(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta F_{\mu\nu}(y) \right] \theta_\beta + 4 \left(\sigma^\nu \frac{\partial \bar{\lambda}(y)}{\partial x^\nu} \right)_\alpha \theta\theta$$

$$V(x, \theta, \bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + i\theta\theta m(x) - i\bar{\theta}\bar{\theta} m^\dagger(x) + \theta\sigma^\mu \bar{\theta} V_\mu$$

$$+ i\theta\theta\bar{\theta} \left(\bar{\lambda}(x) - \frac{1}{2} i\bar{\sigma}^\mu \frac{\partial \chi(x)}{\partial x^\mu} \right) - i\bar{\theta}\bar{\theta}\theta \left(\lambda(x) - \frac{1}{2} i\sigma^\mu \frac{\partial \bar{\chi}(x)}{\partial x^\mu} \right) + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta} \left(D(x) - \frac{1}{2} \frac{\partial^2 C(x)}{\partial x^\mu \partial x_\mu} \right)$$

$$\text{with } D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}, \quad y^\mu = x^\mu - i\theta \sigma^\mu \bar{\theta}$$

である。non-Abelian では、

$$W_\alpha(y) = \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x)) (D_\alpha \exp(V(x)))$$

$$= \frac{1}{2g} \bar{D}^2 \left(D_\alpha V - \frac{1}{2} [V, D_\alpha V] + \frac{1}{3!} [V, [V, D_\alpha V]] + \dots \right)$$

$V = V_{ws} = \theta\sigma^\mu \bar{\theta} V_\mu + i\theta\theta\bar{\lambda} - i\bar{\theta}\bar{\theta}\lambda + \frac{1}{2} \theta\theta\bar{\theta}\bar{\theta} D$ のゲージの時に $V^a V^b V^c = 0$ なので、

$$W_\alpha(y) = \frac{1}{2g} \bar{D}^2 \left(D_\alpha V - \frac{1}{2} [V, D_\alpha V] \right)$$

になる。ここで、第一項は

$$\bar{D}^2 D_\alpha V = 4i\lambda_\alpha(y) - \left[4D(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta F_{\mu\nu}(y) \right] \theta_\beta + 4 \left(\sigma^\nu \frac{\partial \bar{\lambda}(y)}{\partial x^\nu} \right)_\alpha \theta\theta$$

である。第二項は、

$$D_\alpha V = i\chi_\alpha(x) + 2i\theta_\alpha m(x) + (\sigma^\mu \bar{\theta})_\alpha V_\mu + 2i\theta_\alpha \bar{\theta} \left(\bar{\lambda}(x) - \frac{1}{2} i\bar{\sigma}^\mu \frac{\partial \chi(x)}{\partial x^\mu} \right) - i\bar{\theta}\bar{\theta} \left(\lambda(x) - \frac{1}{2} i\sigma^\mu \frac{\partial \bar{\chi}(x)}{\partial x^\mu} \right)_\alpha$$

$$+ \frac{1}{2} 2\theta_\alpha \bar{\theta}\bar{\theta} \left(D(x) - \frac{1}{2} \frac{\partial^2 C(x)}{\partial x^\mu \partial x_\mu} \right)$$

$$- i(\sigma^\nu \bar{\theta})_\alpha \frac{\partial}{\partial x^\nu} \left(C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + i\theta\theta m(x) + \theta\sigma^\mu \bar{\theta} V_\mu + i\theta\theta\bar{\theta} \left(\bar{\lambda}(x) - \frac{1}{2} i\bar{\sigma}^\mu \frac{\partial \chi(x)}{\partial x^\mu} \right) \right)$$

$$\stackrel{\text{WZ gauge}}{=} (\sigma^\mu \bar{\theta})_\alpha V_\mu + 2i\theta_\alpha \bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\lambda_\alpha + \frac{1}{2} 2\theta_\alpha \bar{\theta}\bar{\theta} D - i(\sigma^\nu \bar{\theta})_\alpha \left(\frac{\partial V_\mu}{\partial x^\nu} \theta\sigma^\mu \bar{\theta} + i\theta\theta\bar{\theta} \frac{\partial \bar{\lambda}}{\partial x^\nu} \right)$$

より

$$\begin{aligned}
[V, D_\alpha V] &= \left[\begin{array}{l} \theta \sigma^\mu \bar{\theta} V_\mu + i \theta \theta \bar{\theta} \lambda - i \bar{\theta} \theta \theta \lambda + \frac{1}{2} \theta \theta \bar{\theta} \theta D, \\ (\sigma^\nu \bar{\theta})_\alpha V_\nu + 2i \theta_\alpha \bar{\theta} \lambda - i \bar{\theta} \theta \lambda_\alpha + \frac{1}{2} 2 \theta_\alpha \bar{\theta} \theta D - i (\sigma^\nu \bar{\theta})_\alpha \left(\frac{\partial V_\rho}{\partial x^\nu} \theta \sigma^\rho \bar{\theta} + i \theta \theta \bar{\theta} \frac{\partial \bar{\lambda}}{\partial x^\rho} \right) \end{array} \right] \\
&= \left[\theta \sigma^\mu \bar{\theta} V_\mu + i \theta \theta \bar{\theta} \lambda, (\sigma^\nu \bar{\theta})_\alpha V_\nu + 2i \theta_\alpha \bar{\theta} \lambda \right] = \left[\theta \sigma^\mu \bar{\theta} V_\mu + i \theta \theta \bar{\theta} \lambda, (\sigma^\nu \bar{\theta})_\alpha V_\nu \right] + 2i \left[\theta \sigma^\mu \bar{\theta} V_\mu, \theta_\alpha \bar{\theta} \lambda \right] \\
&= \left[-\bar{\theta}_\beta (\bar{\sigma}^\mu \theta)^\beta V_\mu + i \theta \theta \bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + 2i \left[\theta^\beta (\sigma^\mu)_{\beta\beta'} \bar{\theta}^{\beta'} V_\mu, \bar{\theta}_\beta \bar{\lambda}^{\beta'} \theta_\alpha \right] \\
&= \left[-\bar{\theta}_\beta (\bar{\sigma}^\mu \theta)^\beta V_\mu, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + \left[i \theta \theta \bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + 2i \left[\theta^\beta (\sigma^\mu)_{\beta\beta'} \bar{\theta}^{\beta'} V_\mu, \theta_\alpha \bar{\theta}_\alpha \bar{\lambda}^{\alpha'} \right] \\
&= \bar{\theta}^{\alpha'} \bar{\theta}_\beta (\bar{\sigma}^\mu \theta)^\beta \left[-V_\mu, (\sigma^\nu)_{\alpha\alpha'} V_\nu \right] + i \theta \theta \left[\bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + 2i \theta^\beta \bar{\theta}^{\beta'} \theta_\alpha \bar{\theta}_\alpha \left[(\sigma^\mu)_{\beta\beta'} V_\mu, \bar{\lambda}^{\alpha'} \right] \\
&= -\frac{1}{2} \bar{\theta} \theta (\sigma^\nu)_{\alpha\alpha'} (\bar{\sigma}^\mu \theta)^{\alpha'} \left[-V_\mu, V_\nu \right] + i \theta \theta \left[\bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + 2i \left(\frac{1}{2} \theta \theta \right) \bar{\theta}_\alpha \bar{\theta}^{\beta'} \left[(\sigma^\mu)_{\beta\beta'} V_\mu, \bar{\lambda}^{\beta'} \right] \\
&= \frac{1}{2} \bar{\theta} \theta (\sigma^\nu \bar{\sigma}^\mu \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \left[\bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + 2i \frac{1}{2} \theta \theta \frac{1}{2} \bar{\theta} \theta \left[V_\mu, (\sigma^\mu)_{\alpha\beta'} \bar{\lambda}^{\beta'} \right] \\
&= \frac{1}{2} \bar{\theta} \theta (\sigma^\nu \bar{\sigma}^\mu \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \left[\bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] + \frac{1}{2} i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right] \\
&\Leftarrow \sigma^\mu \bar{\sigma}^\nu = g^{\mu\nu} + i \sigma^{\mu\nu} \\
&\Leftarrow \left[\bar{\theta}_\beta \bar{\lambda}^\beta, (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \right] = \bar{\theta}_\beta \bar{\lambda}^\beta (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu - (\sigma^\nu)_{\alpha\alpha'} \bar{\theta}^{\alpha'} V_\nu \bar{\theta}_\beta \bar{\lambda}^\beta \\
&= -\bar{\theta}_\beta \bar{\theta}^{\alpha'} \bar{\lambda}^\beta (\sigma^\nu)_{\alpha\alpha'} V_\nu + (\sigma^\nu)_{\alpha\alpha'} V_\nu \bar{\theta}_\beta \bar{\theta}^{\alpha'} \bar{\lambda}^\beta = -\frac{1}{2} \bar{\theta} \theta (\sigma^\nu \bar{\lambda})_\alpha V_\nu + (\sigma^\nu \bar{\lambda})_\alpha V_\nu \frac{1}{2} \bar{\theta} \theta \\
&= -\frac{1}{2} \bar{\theta} \theta \left[(\sigma^\nu \bar{\lambda})_\alpha, V_\nu \right] \\
&= \frac{1}{2} \bar{\theta} \theta (i \sigma^{\nu\mu} \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \left(-\frac{1}{2} \bar{\theta} \theta \left[(\sigma^\nu \bar{\lambda})_\alpha, V_\nu \right] \right) + \frac{1}{2} i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right] \\
&= -\frac{1}{2} i \bar{\theta} \theta (\sigma^{\mu\nu} \theta)_\alpha \left[V_\mu, V_\nu \right] + 2 \frac{1}{2} i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right]
\end{aligned}$$

故に、

$$\begin{aligned}
\bar{D}^2 [V, D_\alpha V] &= \left(-\frac{\partial}{\partial \bar{\theta}^\beta} \frac{\partial}{\partial \bar{\theta}_\beta} + 2i (\theta \sigma^\rho)_\beta \frac{\partial}{\partial \bar{\theta}_\beta} \frac{\partial}{\partial x^\rho} + \theta \theta \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} \right) [V, D_\alpha V] \\
&= \left(-\frac{\partial}{\partial \bar{\theta}^\beta} \frac{\partial}{\partial \bar{\theta}_\beta} + 2i (\theta \sigma^\rho)_\beta \frac{\partial}{\partial \bar{\theta}_\beta} \frac{\partial}{\partial x^\rho} + \theta \theta \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} \right) \left(-\frac{1}{2} i \bar{\theta} \theta (\sigma^{\mu\nu} \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right] \right) \\
&= \left(-\frac{\partial}{\partial \bar{\theta}^\beta} \frac{\partial}{\partial \bar{\theta}_\beta} + 2i (\theta \sigma^\rho)_\beta \frac{\partial}{\partial \bar{\theta}_\beta} \frac{\partial}{\partial x^\rho} \right) \left(-\frac{1}{2} i \bar{\theta} \theta (\sigma^{\mu\nu} \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right] \right) \\
&= \left(-\frac{\partial}{\partial \bar{\theta}^\beta} \frac{\partial}{\partial \bar{\theta}_\beta} \right) \left(-\frac{1}{2} i \bar{\theta} \theta (\sigma^{\mu\nu} \theta)_\alpha \left[V_\mu, V_\nu \right] + i \theta \theta \bar{\theta} \theta \left[V_\mu, (\sigma^\mu \bar{\lambda})_\alpha \right] \right) \\
&\quad + \left(2i (\theta \sigma^\rho)_\beta \frac{\partial}{\partial \bar{\theta}_\beta} \frac{\partial}{\partial x^\rho} \right) \left(-\frac{1}{2} i \bar{\theta} \theta (\sigma^{\mu\nu} \theta)_\alpha \left[V_\mu, V_\nu \right] \right) \Leftarrow \frac{\partial}{\partial \bar{\theta}^\alpha} \frac{\partial}{\partial \bar{\theta}_\alpha} \bar{\theta} \theta = 4
\end{aligned}$$

$$\begin{aligned}
&= (-) \left(-\frac{1}{2} i 4 (\sigma^{\mu\nu} \theta)_\alpha [V_\mu, V_\nu] + i \theta \theta 4 [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \right) + 2i (\theta \sigma^\rho)_\beta \left(-\frac{1}{2} i 2 \bar{\theta}^\beta (\sigma^{\mu\nu} \theta)_\alpha \frac{\partial}{\partial x^\rho} [V_\mu, V_\nu] \right) \\
&= (-) \left(-\frac{1}{2} i 4 \right) \left((\sigma^{\mu\nu} \theta)_\alpha [V_\mu, V_\nu] - 2 \theta \theta [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \right) + (\sigma^{\mu\nu} \theta)_\alpha 2i (\theta \sigma^\rho)_\beta \left(-\frac{1}{2} i 2 \right) \bar{\theta}^\beta \frac{\partial}{\partial x^\rho} [V_\mu, V_\nu] \\
&= 2i \left((\sigma^{\mu\nu} \theta)_\alpha [V_\mu, V_\nu] - 2 \theta \theta [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \right) + (\sigma^{\mu\nu} \theta)_\alpha 2i (\theta \sigma^\rho \bar{\theta}) \left(-\frac{1}{2} i 2 \right) \frac{\partial}{\partial x^\rho} [V_\mu, V_\nu] \\
&= 2i \left((\sigma^{\mu\nu} \theta)_\alpha [V_\mu, V_\nu] - 2 \theta \theta [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \right) + 2 (\sigma^{\mu\nu} \theta)_\alpha (\theta \sigma^\rho \bar{\theta}) \frac{\partial}{\partial x^\rho} [V_\mu, V_\nu] \\
&= 2i (\sigma^{\mu\nu} \theta)_\alpha \left(1 - i (\theta \sigma^\rho \bar{\theta}) \frac{\partial}{\partial x^\rho} \right) [V_\mu, V_\nu] - 4i \theta \theta [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \\
&\Leftarrow (\sigma^{\mu\nu} \theta)_\alpha \left(1 - i (\theta \sigma^\rho \bar{\theta}) \frac{\partial}{\partial x^\rho} \right) f_{\mu\nu}(x) = (\sigma^{\mu\nu} \theta)_\alpha f_{\mu\nu}(y) \quad \text{with } f_{\mu\nu} = [V_\mu, V_\nu] \text{ \& } y = x - i \theta \sigma \bar{\theta} \\
&= 2i (\sigma^{\mu\nu} \theta)_\alpha f_{\mu\nu}(y) - 4i \theta \theta [V_\mu(y), (\sigma^\mu \bar{\lambda}(y))_\alpha] \\
&\Rightarrow \bar{D}^2 [V, D_\alpha V] = 2i (\sigma^{\mu\nu} \theta)_\alpha f_{\mu\nu}(y) - 4i \theta \theta [V_\mu(y), (\sigma^\mu \bar{\lambda}(y))_\alpha]
\end{aligned}$$

になる。これより、

$$\begin{aligned}
2gW_\alpha(y) &= \bar{D}^2 \left(D_\alpha V - \frac{1}{2} [V, D_\alpha V] \right) = 4i \lambda_\alpha(y) - \left[4D(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta F_{\mu\nu}(y) \right] \theta_\beta + 4 \left(\sigma^\nu \frac{\partial \bar{\lambda}(y)}{\partial x^\nu} \right)_\alpha \theta \theta \\
&\quad - \frac{1}{2} \left(2i (\sigma^{\mu\nu} \theta)_\alpha f_{\mu\nu}(y) - 4i \theta \theta [V_\mu(y), (\sigma^\mu \bar{\lambda}(y))_\alpha] \right) \\
&= 4i \lambda_\alpha(y) - \left[4D(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta \left(F_{\mu\nu}(y) + \frac{1}{2} i f_{\mu\nu}(y) \right) \right] \theta_\beta + 4 \left(\frac{\partial}{\partial x^\mu} (\sigma^\mu \bar{\lambda}(y))_\alpha + \frac{1}{2} i [V_\mu, (\sigma^\mu \bar{\lambda})_\alpha] \right) \theta \theta
\end{aligned}$$

ここに、通常の変微分として、

$$\mathcal{D}_\mu \equiv \partial_\mu + \frac{1}{2} i V_\mu, \quad V_{\mu\nu}(y) \equiv F_{\mu\nu}(y) + \frac{1}{2} i f_{\mu\nu}(y) = \partial_\mu V_\nu - \partial_\nu V_\mu + \frac{1}{2} i [V_\mu, V_\nu] = \mathcal{D}_\mu V_\nu - \mathcal{D}_\nu V_\mu$$

に注意すると、

$$2gW_\alpha(y) = \bar{D}^2 \left(D_\alpha V - \frac{1}{2} [V, D_\alpha V] \right) = 4i \lambda_\alpha(y) - \left[4D(y) \delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}(y) \right] \theta_\beta + 4 \mathcal{D}_\mu (\sigma^\mu \bar{\lambda}(y))_\alpha \theta \theta$$

が求まる。成分表示すれば、 $V = 2g \sum_{a=1}^N V^a t^a$ に注意して、

$$\begin{aligned}
\mathcal{D}_\mu &= \partial_\mu + \frac{1}{2} i V_\mu = \partial_\mu + \frac{1}{2} i 2g \sum_{a=1}^N V_\mu^a t^a = \partial_\mu + i g \sum_{a=1}^N V_\mu^a t^a \\
V_{\mu\nu}(y) &= 2g \sum_{a=1}^N V_{\mu\nu}^a t^a = 2g \sum_{a=1}^N (\partial_\mu V_\nu^a t^a - \partial_\nu V_\mu^a t^a) + \frac{1}{2} i \left[2g \sum_{b=1}^N V_\mu^b t^b, 2g \sum_{c=1}^N V_\nu^c t^c \right] \\
&\Rightarrow \sum_{a=1}^N V_{\mu\nu}^a t^a = \sum_{a=1}^N (\partial_\mu V_\nu^a t^a - \partial_\nu V_\mu^a t^a) + i g \left[\sum_{b=1}^N V_\mu^b t^b, \sum_{c=1}^N V_\nu^c t^c \right] \Leftarrow [t^b, t^c] = i f^{bca} t^a = i f^{abc} t^a \\
V_{\mu\nu}^a t^a &= \partial_\mu V_\nu^a t^a - \partial_\nu V_\mu^a t^a + i g i f^{abc} t^a V_\mu^b V_\nu^c = \partial_\mu V_\nu^a t^a - \partial_\nu V_\mu^a t^a - g f^{abc} t^a V_\mu^b V_\nu^c \\
&\Rightarrow V_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - g f^{abc} V_\mu^b V_\nu^c
\end{aligned}$$

がわかる。つまり、

$$\begin{aligned}
2gW_\alpha(y) &= 4i\lambda_\alpha(y) - \left[4D(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}(y) \right] \theta_\beta + 4\mathcal{D}_\mu(\sigma^\mu \bar{\lambda}(y))_\alpha \theta\theta \\
&= 2g \sum_a \left\{ 4i\lambda_\alpha^a(y) - \left[4D^a(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4\mathcal{D}_\mu(\sigma^\mu \bar{\lambda}^a(y))_\alpha \theta\theta \right\} t^a \\
\Rightarrow W_\alpha(y) &= 4i\lambda_\alpha(y) - \left[4D(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}(y) \right] \theta_\beta + 4\mathcal{D}_\mu(\sigma^\mu \bar{\lambda}(y))_\alpha \theta\theta \\
&= \left(4i\lambda_\alpha^a(y) - \left[4D^a(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4\mathcal{D}_\mu(\sigma^\mu \bar{\lambda}^a(y))_\alpha \theta\theta \right) t^a
\end{aligned}$$

以上から、

$$\begin{aligned}
W_\alpha(y) &= \frac{1}{2ig} \bar{D}_\beta \bar{D}^\beta \mathbf{G}_\alpha(x) = \frac{1}{2g} \bar{D}_\beta \bar{D}^\beta \exp(-V(x)) (D_\alpha \exp(V(x))) \\
&\Leftarrow V_{WS} = \theta\sigma^\mu \bar{\theta} V_\mu + i\theta\theta \bar{\theta} \bar{\lambda} - i\bar{\theta}\theta \theta \lambda + \frac{1}{2} \theta\theta \bar{\theta} \bar{\theta} D \\
W_\alpha(y) &= \frac{1}{2g} \bar{D}^2 \left(D_\alpha V_{WS} - \frac{1}{2} [V_{WS}, D_\alpha V_{WS}] \right) \\
&= \left(4i\lambda_\alpha^a(y) - \left[4D^a(y)\delta_\alpha^\beta + 2(\sigma^{\mu\nu})_{\bar{\alpha}}^\beta V_{\mu\nu}^a(y) \right] \theta_\beta + 4(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}^a(y))_\alpha \theta\theta \right) t^a \\
&\Leftarrow \mathcal{D}_\mu = \partial_\mu + \frac{1}{2} iV_\mu = \partial_\mu + ig \sum_{a=1}^N V_\mu^a t^a, \quad V_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - g f^{abc} V_\mu^b V_\nu^c
\end{aligned}$$

が得られた。

R 対称性

超対称演算子へのカイラル変換を導入する。

$$\begin{aligned}
U^\dagger S_\alpha U &= \Omega S_\alpha \stackrel{\text{def}}{=} \exp(-i\alpha) S_\alpha \text{ with } U = \exp(i\alpha R) \text{ and } \Omega = \exp(i\alpha \tilde{R}) \\
\Rightarrow \delta_R S_\alpha &= [i\alpha R, S_\alpha] = -i\alpha \tilde{R}_S S_\alpha = -i\alpha S_\alpha \Leftrightarrow \delta_R \bar{S}_\alpha = [i\alpha R, \bar{S}_\alpha] = i\alpha \tilde{R}_S \bar{S}_\alpha = i\alpha \bar{S}_\alpha
\end{aligned}$$

これを *R-charge* と呼ぶ。そのとき、それらの表現である微分演算子の間に、

$$[\tilde{R}_S, \tilde{S}_\alpha] = -\tilde{S}_\alpha \text{ with } \tilde{S}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, [\tilde{R}_S, \tilde{\bar{S}}_\alpha] = \tilde{\bar{S}}_\alpha \text{ with } \tilde{\bar{S}}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}$$

が成立する。この解は、

$$\begin{aligned}
[\tilde{R}_S, \tilde{S}_\alpha] &= -\tilde{S}_\alpha \Rightarrow \tilde{R}_S \left(\frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \right) - \left(\frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \right) \tilde{R}_S = - \left(\frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \right) \\
\Rightarrow \tilde{R}_S \frac{\partial f}{\partial \theta^\alpha} - \frac{\partial}{\partial \theta^\alpha} (\tilde{R}_S f) &= \tilde{R}_S \frac{\partial f}{\partial \theta^\alpha} - \frac{\partial \tilde{R}_S}{\partial \theta^\alpha} f - \tilde{R}_S \frac{\partial f}{\partial \theta^\alpha} = - \frac{\partial f}{\partial \theta^\alpha} \\
\rightarrow - \frac{\partial \tilde{R}_S}{\partial \theta^\alpha} f &= - \frac{\partial f}{\partial \theta^\alpha} \rightarrow \frac{\partial \tilde{R}_S}{\partial \theta^\alpha} = \frac{\partial}{\partial \theta^\alpha} \rightarrow \tilde{R}_S = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \\
\Rightarrow \tilde{R}_S \left(i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial f}{\partial x^\mu} \right) &- i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} (\tilde{R}_S f) \\
&= i(\tilde{R}_S \sigma^\mu \bar{\theta})_\alpha \frac{\partial f}{\partial x^\mu} + i(\sigma^\mu \bar{\theta})_\alpha \tilde{R}_S \frac{\partial f}{\partial x^\mu} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial \tilde{R}_S}{\partial x^\mu} f - i(\sigma^\mu \bar{\theta})_\alpha \tilde{R}_S \frac{\partial f}{\partial x^\mu} = -i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial f}{\partial x^\mu} \\
\rightarrow i(\tilde{R}_S \sigma^\mu \bar{\theta})_\alpha \frac{\partial f}{\partial x^\mu} &- i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial \tilde{R}_S}{\partial x^\mu} f = -i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial f}{\partial x^\mu} \rightarrow \tilde{R}_S = -\bar{\theta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha}
\end{aligned}$$

以上から、

$$\tilde{R}_S = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} - \bar{\theta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha}$$

\tilde{S}_α についても同様に、同じ演算子が得られる。添え字の上げ下げは「第3章超空間」の

$$\theta_\alpha \frac{\partial}{\partial \theta_\alpha} = \theta^\alpha \frac{\partial}{\partial \theta^\alpha}, \quad \bar{\theta}_\alpha \frac{\partial}{\partial \bar{\theta}_\alpha} = \bar{\theta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha}$$

に注意する。更に、

$$\tilde{R}_S \theta = \theta \quad \text{and} \quad \tilde{R}_S \bar{\theta} = -\bar{\theta}$$

なので、

$$\begin{aligned} \theta' &\equiv \exp(i\alpha \tilde{R}_S) \theta = \sum_n \frac{1}{n!} (i\alpha \tilde{R}_S)^n \theta \leftarrow \tilde{R}_S \theta = \theta \rightarrow (\tilde{R}_S)^n \theta = \theta \\ &= \sum_n \frac{(i\alpha)^n}{n!} \theta = \exp(i\alpha) \theta \end{aligned}$$

より

$$\theta' = \exp(i\alpha \tilde{R}_S) \theta = e^{i\alpha} \theta \quad \text{and} \quad \bar{\theta}' = \exp(i\alpha \tilde{R}_S) \bar{\theta} = e^{-i\alpha} \bar{\theta}$$

がわかる。

さて、任意の場 $\hat{\phi}(x, \theta, \bar{\theta})$ の演算子に対して R -charge を n とするとき

$$\hat{\phi}'(x, \theta', \bar{\theta}') = \exp(-i\alpha R) \hat{\phi}(x, \theta, \bar{\theta}) \exp(i\alpha R) \stackrel{\text{def}}{=} \exp(i\alpha n) \hat{\phi}(x, \theta, \bar{\theta})$$

と定義する。その時、

$$\hat{\phi}'(x, \theta', \bar{\theta}') = \hat{\phi}'(x, e^{i\alpha} \theta, e^{-i\alpha} \bar{\theta}) = \exp(i\alpha \tilde{R}_S) \hat{\phi}'(x, \theta, \bar{\theta})$$

に注意すると、

$$\begin{aligned} \hat{\phi}'(x, \theta', \bar{\theta}') \exp(-i\alpha \tilde{R}_S) \exp(i\alpha n) \hat{\phi}(x, \theta, \bar{\theta}) &= \exp(-i\alpha n) \hat{\phi}(x, \theta, \bar{\theta}) \\ \Rightarrow \hat{\phi}'(x, \theta, \bar{\theta}) &= \exp(-i\alpha \tilde{R}_S) \exp(i\alpha n) \hat{\phi}(x, \theta, \bar{\theta}) \stackrel{\text{def}}{=} \exp(i\alpha \tilde{R}_\varphi) \hat{\phi}(x, \theta, \bar{\theta}) \end{aligned}$$

で導入される \tilde{R}_φ は、

$$\tilde{R}_\varphi = n - \tilde{R}_S \quad \text{with} \quad \tilde{R}_S = \theta^\alpha \frac{\partial}{\partial \theta^\alpha} - \bar{\theta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha}$$

になる。これを成分場の R -charge として定義できる。従って、

1) カイラル超場

$$\begin{aligned}
\Phi'(y, \theta) &= \phi'(y) - \sqrt{2}\theta\psi'(y) + F'(y)\theta\theta \\
&= \exp(i\alpha\tilde{R}_\phi)\Phi(y, \theta) = \exp(i\alpha\tilde{R}_\phi)\left(\phi(y) - \sqrt{2}\theta\psi(y) + F(y)\theta\theta\right) \\
&= \exp(i\alpha n_\phi - i\alpha\tilde{R}_S)\left(\phi(y) - \sqrt{2}\theta\psi(y) + F(y)\theta\theta\right) \\
&= e^{i\alpha n_\phi}\left(\phi(y) - \sqrt{2}e^{-i\alpha}\theta\psi(y) + F(y)e^{-2i\alpha}\theta\theta\right) \\
\Rightarrow \phi'(y) &= e^{i\alpha n_\phi}\phi(y), \quad \psi'(y) = e^{i\alpha(n_\phi-1)}\psi(y), \quad F'(y) = e^{i\alpha(n_\phi-2)}F(y)
\end{aligned}$$

2) ゲージ超場

$$\begin{aligned}
V'_{WZ}(x, \theta, \bar{\theta}) &= \theta\sigma^\mu\bar{\theta}V'_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}'(x) - i\bar{\theta}\bar{\theta}\theta\lambda'(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D'(x) \\
&= \exp(i\alpha\tilde{R}_\phi)V_{WZ}(x, \theta, \bar{\theta}) = \exp(i\alpha\tilde{R}_\phi)\left(\theta\sigma^\mu\bar{\theta}V_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\right) \\
&= \exp(i\alpha n_\nu - i\alpha\tilde{R}_S)\left(\theta\sigma^\mu\bar{\theta}V_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\right) \\
&= e^{i\alpha n_\nu}\left(\theta\sigma^\mu\bar{\theta}V_\mu(x) + ie^{-i\alpha}\theta\theta\bar{\theta}\bar{\lambda}(x) - ie^{i\alpha}\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\right) \\
\Rightarrow V'_\mu(x) &= e^{i\alpha n_\nu}V_\mu(x), \quad \bar{\lambda}'(x) = e^{i\alpha(n_\nu-1)}\bar{\lambda}(x), \quad \lambda'(x) = e^{i\alpha(n_\nu+1)}\lambda(x), \quad D'(x) = e^{i\alpha n_\nu}D(x)
\end{aligned}$$

以上から

$ \begin{aligned} R(\phi) &= n_\phi; \quad R(\psi) = n_\phi - 1; \quad R(F) = n_\phi - 2 \\ R(V_\mu) &= R(D) = n_\nu; \quad R(\lambda) = n_\nu + 1 \end{aligned} $
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を得る。