

第4章 超空間

超対称性代数

$$\begin{aligned} [S_\alpha, P_\mu] &= [\bar{S}_\alpha, P_\mu] = [P_\mu, P_\nu] = \{S_\alpha, S_\beta\} = \{\bar{S}_\alpha, \bar{S}_\beta\} = 0 \\ \{S_\alpha, \bar{S}_\beta\} &= 2(\sigma^\mu)_{\alpha\beta} P_\mu, \quad \{S^\alpha, \bar{S}^\beta\} = -2(\bar{\sigma}^\mu)^{\alpha\beta} P_\mu \end{aligned}$$

この代数から得られる微小並進変換は、

$$\delta A = [\varepsilon S + \bar{\varepsilon} \bar{S} + ia^\mu P_\mu, A]$$

で定義されている。一方、「**場の変換性と演算子**」

$$\begin{aligned} \hat{\phi}(X') &= U \hat{\phi}(X) U^\dagger = \tilde{U} \hat{\phi}(X) \text{ with } X' = X + A \\ \text{with } U &= \exp(-iT(a)) \quad \& \quad \tilde{U} = \exp(it(a)) \\ T(a) &= \sum_A a^A Q^A \left(t(a) = \sum_A a^A \tilde{Q}^A \right) \text{ with } T^\dagger(a) = T(a) \left(t^\dagger(a) = t(a) \right) \\ \exp(-iT(a)) \hat{\phi}(X) \exp(iT(a)) &= \exp(it(a)) \hat{\phi}(X) = \hat{\phi}(X + A) \\ \delta \hat{\phi}(X) &= [iT(a), \hat{\phi}(X)] = -it(a) \hat{\phi}(X) = -A^\alpha \frac{\partial \hat{\phi}(X)}{\partial X^\alpha} \end{aligned}$$

のまとめより超対称性の変換は

$$\begin{aligned} i \sum_A a^A Q^A &\equiv \varepsilon S + \bar{\varepsilon} \bar{S} \Leftrightarrow \delta \hat{\phi}(X) = \left[i \sum_A a^A Q^A, \hat{\phi}(X) \right] = \left[\varepsilon S + \bar{\varepsilon} \bar{S}, \hat{\phi}(X) \right] \\ \Rightarrow \left[\varepsilon S + \bar{\varepsilon} \bar{S}, \hat{\phi}(X) \right] &= - \left(\varepsilon^\alpha \frac{\partial \hat{\phi}(X)}{\partial \theta^\alpha} + \bar{\varepsilon}_\alpha \frac{\partial \hat{\phi}(X)}{\partial \bar{\theta}_\alpha} \right) \quad (\text{for superspace of } \theta \& \bar{\theta}) \end{aligned}$$

で与えられるので、 $i \sum_A a^A Q^A = ia^\mu P_\mu$ と組み合わせて

$$U = \exp \left(-i \sum_A a^A Q^A \right) = \exp \left(-\varepsilon S - \bar{\varepsilon} \bar{S} - ia^\mu P_\mu \right)$$

これを G で表し

$$G(x^\mu, \varepsilon, \bar{\varepsilon}) = \exp \left(-\varepsilon S - \bar{\varepsilon} \bar{S} - ia^\mu P_\mu \right)$$

この変換 G は群をなす。つまり、

$$G(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) = G(x'^\mu, \theta', \bar{\theta}')$$

である。これを証明する。よく知られた公式

$$\exp(X) \exp(Y) = \exp \left(X + Y + \frac{1}{2} [X, Y] \right)$$

を利用する。ただし、条件式

$$[X, [X, Y]] = [Y, [X, Y]] = 0$$

を満たしている必要がある。

$$X = -\theta S - \bar{\theta} \bar{S} - ix^\mu P_\mu, \quad Y = -\xi S - \bar{\xi} \bar{S} - ia^\mu P_\mu$$

なので、条件式を

$$[X, X+Y] \propto 0 \text{ or } P_\mu, \quad [Y, X+Y] \propto 0 \text{ or } P_\mu$$

満たしている。そこで、

$$\begin{aligned} & \exp(-\xi S - \bar{\xi} \bar{S} - ia^\mu P_\mu) \exp(-\theta S - \bar{\theta} \bar{S} - ix^\mu P_\mu) \\ &= \exp\left(-\xi S - \bar{\xi} \bar{S} - ia^\mu P_\mu - \theta S - \bar{\theta} \bar{S} - ix^\mu P_\mu + \frac{1}{2}[-\xi S - \bar{\xi} \bar{S} - ia^\mu P_\mu, -\theta S - \bar{\theta} \bar{S} - ix^\mu P_\mu]\right) \\ &= \exp\left(-(\theta + \xi)S - (\bar{\theta} + \bar{\xi})\bar{S} - i(x+a)^\mu P_\mu + \frac{1}{2}[\xi S + \bar{\xi} \bar{S} + ia^\mu P_\mu, \theta S + \bar{\theta} \bar{S} + ix^\mu P_\mu]\right) \\ &\quad \Leftarrow [\xi S + \bar{\xi} \bar{S} + ia^\mu P_\mu, \theta S + \bar{\theta} \bar{S} + ix^\mu P_\mu] = [\xi S, \bar{\theta} \bar{S}] + [\bar{\xi} \bar{S}, \theta S] = [\xi S, \bar{\theta} \bar{S}] - [\theta S, \bar{\xi} \bar{S}] \\ &\quad = 2\xi\sigma^\mu \bar{\theta} P_\mu - 2\theta\sigma^\mu \bar{\xi} P_\mu \Leftarrow [\varepsilon_1 S, \bar{\varepsilon}_2 \bar{S}] = 2\varepsilon_1\sigma^\mu \bar{\varepsilon}_2 P_\mu \\ &= \exp\left(-(\theta + \xi)S - (\bar{\theta} + \bar{\xi})\bar{S} - i(x+a)^\mu P_\mu + \frac{1}{2}(2\xi\sigma^\mu \bar{\theta} P_\mu - 2\theta\sigma^\mu \bar{\xi} P_\mu)\right) \\ &= \exp\left(-(\theta + \xi)S - (\bar{\theta} + \bar{\xi})\bar{S} - i(x+a)^\mu P_\mu + \xi\sigma^\mu \bar{\theta} P_\mu - \theta\sigma^\mu \bar{\xi} P_\mu\right) \\ &= \exp\left(-(\theta + \xi)S - (\bar{\theta} + \bar{\xi})\bar{S} - i(x^\mu + a^\mu + i\xi\sigma^\mu \bar{\theta} - i\theta\sigma^\mu \bar{\xi})P_\mu\right) \\ &\equiv \exp(-\theta' S - \bar{\theta}' \bar{S} - ix'^\mu P_\mu) \end{aligned}$$

つまり

$$G(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) = G(x^\mu + a^\mu + i\xi\sigma^\mu \bar{\theta} - i\theta\sigma^\mu \bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi})$$

であることがわかる。一方、 $(x, \theta, \bar{\theta})$ に依存する超場 $\Phi(x, \theta, \bar{\theta})$ を導入すれば、

$$\begin{aligned} \hat{\phi}(X') &= U \hat{\phi}(X) U^\dagger = \tilde{U} \hat{\phi}(X) \text{ with } X' = X + A \\ \text{with } U &= \exp(-iT(a)) \quad \& \quad \tilde{U} = \exp(it(a)) = \exp\left(A^\alpha \frac{\partial}{\partial X^\alpha}\right) \end{aligned}$$

を参考にして $G(x^\mu, \varepsilon, \bar{\varepsilon}) = \exp(-\varepsilon S - \bar{\varepsilon} \bar{S} - ia^\mu P_\mu)$ なので

$$\begin{aligned} G(a^\mu, \xi, \bar{\xi}) \Phi(x, \theta, \bar{\theta}) G^\dagger(a^\mu, \xi, \bar{\xi}) &= \Phi(x^\mu + a^\mu + i\xi\sigma^\mu \bar{\theta} - i\theta\sigma^\mu \bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) \\ &= \tilde{G}(a^\mu, \xi, \bar{\xi}) \Phi(x^\mu, \theta, \bar{\theta}) \end{aligned}$$

の $\tilde{G}(x^\mu, \varepsilon, \bar{\varepsilon})$ は $\tilde{G}(x^\mu, \varepsilon, \bar{\varepsilon}) = \exp(\varepsilon \tilde{S} + \bar{\varepsilon} \tilde{\bar{S}} + ia^\mu \tilde{P}_\mu)$ となるので (S, \bar{S}, P_μ) 対応する generators は

$$\tilde{G}(x^\mu, \theta, \bar{\theta}) = \exp(\theta \tilde{S} + \bar{\theta} \tilde{\bar{S}} + ix^\mu \tilde{P}_\mu)$$

で表される。ところで、

$$\exp(aA) = \lim_{n \rightarrow \infty} \left(1 + a \frac{A}{n}\right)^n$$

なので、 A 自身がわかればよい。そこで、 $|a| \ll 1$ の微小変換

$$\exp(aA) \approx 1 + aA$$

を利用すれば A がわかる。

$$\Phi(x^\mu + a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{G}(a^\mu, \xi, \bar{\xi})\Phi(x^\mu, \theta, \bar{\theta})$$

の式の両辺で微小変換をとり、 $(\xi, \bar{\xi}, a)$ の一次の項のみとり二次以降を無視する：

$$\begin{aligned} \Phi(x^\mu + a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) &\cong \Phi + (a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}) \frac{\partial\Phi}{\partial x^\mu} + \xi^\alpha \frac{\partial\Phi}{\partial\theta^\alpha} + \bar{\xi}_\alpha \frac{\partial\Phi}{\partial\bar{\theta}_\alpha} \\ \tilde{G}(a^\mu, \xi, \bar{\xi})\Phi(x^\mu, \theta, \bar{\theta}) &= \exp(\xi\tilde{S} + \bar{\xi}\bar{\tilde{S}} + ia^\mu\tilde{P}_\mu)\Phi \cong (1 + \xi^\alpha\tilde{S}_\alpha + \bar{\xi}_\alpha\bar{\tilde{S}}^\alpha + ia^\mu\tilde{P}_\mu)\Phi \end{aligned}$$

両辺を比較すれば、

$$\boxed{\xi_\alpha \frac{\partial}{\partial\theta_\alpha} = \xi^\alpha \frac{\partial}{\partial\theta^\alpha}, \quad \bar{\xi}_\alpha \frac{\partial}{\partial\bar{\theta}_\alpha} = \bar{\xi}^\alpha \frac{\partial}{\partial\bar{\theta}^\alpha}}$$

$$\begin{aligned} (*) \xi_\alpha \frac{\partial}{\partial\theta_\alpha} \chi\theta &= \xi_\alpha \frac{\partial}{\partial\theta_\alpha} \chi^\beta\theta_\beta = \xi_\alpha \frac{\partial}{\partial\theta_\alpha} (-\theta_\beta\chi^\beta) = -\xi_\alpha\chi^\alpha = \chi^\alpha\xi_\alpha \quad \& \quad \xi^\alpha \frac{\partial}{\partial\theta^\alpha} \chi\theta = \xi^\alpha \frac{\partial}{\partial\theta^\alpha} \theta^\beta\chi_\beta = \xi^\alpha\chi_\alpha = \chi^\alpha\xi_\alpha \\ \xi_\alpha \frac{\partial}{\partial\theta_\alpha} \theta\theta &= \xi_\alpha \frac{\partial}{\partial\theta_\alpha} (-2\theta_1\theta_2) = -2\xi_1\theta_2 + 2\xi_2\theta_1 = -2\varepsilon_{12}\xi^2\theta_2 + 2\varepsilon_{21}\xi^1\theta_1 \\ &= 2\xi^2\theta_2 + 2\xi^1\theta_1 = 2\xi\theta \leftarrow \varepsilon_{12} = -1, \quad \varepsilon_{21} = 1 \\ \xi^\alpha \frac{\partial}{\partial\theta^\alpha} \theta\theta &= \xi^\alpha \frac{\partial}{\partial\theta^\alpha} (-2\theta^1\theta^2) = -2\xi^1\theta^2 + 2\xi^2\theta^1 = -2\varepsilon^{21}\xi^1\theta_1 + 2\varepsilon^{12}\xi^2\theta_2 \\ &= 2\xi^1\theta_1 + 2\xi^2\theta_2 = 2\xi\theta \leftarrow \varepsilon^{21} = -1, \quad \varepsilon^{12} = 1 \end{aligned}$$

に注意して

$$\begin{aligned} &\Phi + (a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}) \frac{\partial\Phi}{\partial x^\mu} + \xi^\alpha \frac{\partial\Phi}{\partial\theta^\alpha} + \bar{\xi}_\alpha \frac{\partial\Phi}{\partial\bar{\theta}_\alpha} \\ &= \Phi + (a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}) \frac{\partial\Phi}{\partial x^\mu} + \xi^\alpha \frac{\partial\Phi}{\partial\theta^\alpha} + \bar{\xi}_\alpha \frac{\partial\Phi}{\partial\bar{\theta}_\alpha} = (1 + \xi\tilde{S} + \bar{\xi}\bar{\tilde{S}} + ia^\mu\tilde{P}_\mu)\Phi \\ &\rightarrow \xi^\alpha\tilde{S}_\alpha\Phi = i\xi^\alpha(\sigma^\mu\bar{\theta})_\alpha \frac{\partial\Phi}{\partial x^\mu} + \xi^\alpha \frac{\partial\Phi}{\partial\theta^\alpha} \rightarrow \tilde{S}_\alpha = i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial\theta^\alpha} \\ &\rightarrow \bar{\xi}_\alpha\bar{\tilde{S}}^\alpha\Phi = -\bar{\xi}_\alpha\bar{\tilde{S}}^\alpha\Phi = -i(\theta\sigma^\mu)_\alpha \bar{\xi}_\alpha \frac{\partial\Phi}{\partial x^\mu} + \bar{\xi}_\alpha \frac{\partial\Phi}{\partial\bar{\theta}_\alpha} \rightarrow -\bar{\tilde{S}}_\alpha = i(\theta\sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial\bar{\theta}_\alpha} \\ &\rightarrow ia^\mu\tilde{P}_\mu\Phi = a^\mu \frac{\partial\Phi}{\partial x^\mu} \rightarrow i\tilde{P}_\mu = \frac{\partial}{\partial x^\mu} \end{aligned}$$

以上から、

$$\begin{aligned}
\tilde{S}_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{S}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}, \quad i\tilde{P}_\mu = \frac{\partial}{\partial x^\mu} \\
\xi \tilde{S} &= i\xi \sigma^\mu \bar{\theta} \frac{\partial}{\partial x^\mu} + \xi^\alpha \frac{\partial}{\partial \theta^\alpha}, \quad \bar{\xi} \bar{S} = -i\theta \sigma^\mu \bar{\xi} \frac{\partial}{\partial x^\mu} + \bar{\xi}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha} \\
\delta \hat{\phi}(X) &= [\varepsilon S + \bar{\varepsilon} \bar{S} + ia^\mu P_\mu, \hat{\phi}(X)] = -(\varepsilon \tilde{S} + \bar{\varepsilon} \bar{S} + ia^\mu \tilde{P}_\mu) \hat{\phi}(X) \\
[\varepsilon_1 S, \bar{\varepsilon}_2 \bar{S}] &= 2\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 P_\mu \quad \& \quad [\varepsilon_1 \tilde{S}, \bar{\varepsilon}_2 \bar{S}] = 2\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \tilde{P}_\mu
\end{aligned}$$

が超対称変換の微分表現になる。

共変微分

この変換に関する共変微分を定義する。共変微分を \tilde{L} で表すと、

$$\begin{aligned}
G(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) &= G(x^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
G(x^\mu, \theta, \bar{\theta}) G(a^\mu, \xi, \bar{\xi}) &= G(x^\mu + a^\mu + i\theta \sigma^\mu \bar{\xi} - i\xi \sigma^\mu \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{L}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta})
\end{aligned}$$

で定義される。その結果、

$$\begin{aligned}
&\tilde{L}(b^\mu, \rho, \bar{\rho}) \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&\quad \Leftarrow G(x^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&\quad \Leftarrow x'^\mu = x^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi}, \quad \theta' = \theta + \xi, \quad \bar{\theta}' = \bar{\theta} + \bar{\xi} \\
&= \tilde{L}(b^\mu, \rho, \bar{\rho}) G(x'^\mu, \theta', \bar{\theta}') \\
&\quad \Leftarrow G(x^\mu + a^\mu + i\theta \sigma^\mu \bar{\xi} - i\xi \sigma^\mu \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{L}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&= G(x'^\mu + b^\mu + i\theta' \sigma^\mu \bar{\rho} - i\rho \sigma^\mu \bar{\theta}', \theta' + \rho, \bar{\theta}' + \bar{\rho}) \\
&\quad \Leftarrow x''^\mu = x'^\mu + b^\mu + i\theta' \sigma^\mu \bar{\rho} - i\rho \sigma^\mu \bar{\theta}' = x^\mu + a^\mu + b^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi} + i(\theta + \xi) \sigma^\mu \bar{\rho} - i\rho \sigma^\mu (\bar{\theta} + \bar{\xi}) \\
&= G(x^\mu + a^\mu + b^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi} + i(\theta + \xi) \sigma^\mu \bar{\rho} - i\rho \sigma^\mu (\bar{\theta} + \bar{\xi}), \theta' + \rho, \bar{\theta}' + \bar{\rho}) \\
&\tilde{G}(a^\mu, \xi, \bar{\xi}) \tilde{L}(b^\mu, \rho, \bar{\rho}) G(x^\mu, \theta, \bar{\theta}) \\
&\quad \Leftarrow G(x^\mu + a^\mu + i\theta \sigma^\mu \bar{\xi} - i\xi \sigma^\mu \bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{L}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&\quad \Leftarrow x''^\mu = x^\mu + b^\mu + i\theta \sigma^\mu \bar{\rho} - i\rho \sigma^\mu \bar{\theta}, \quad \theta'' = \theta + \rho, \quad \bar{\theta}'' = \bar{\theta} + \bar{\rho} \\
&= \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x''^\mu, \theta'', \bar{\theta}'') \\
&\quad \Leftarrow G(x^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta} - i\theta \sigma^\mu \bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) = \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&= G(x''^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta}' - i\theta' \sigma^\mu \bar{\xi}, \theta'' + \xi, \bar{\theta}'' + \bar{\xi}) \\
&\quad \Leftarrow x'''^\mu = x''^\mu + a^\mu + i\xi \sigma^\mu \bar{\theta}' - i\theta' \sigma^\mu \bar{\xi} = x^\mu + a^\mu + b^\mu + i\theta \sigma^\mu \bar{\rho} - i\rho \sigma^\mu \bar{\theta} + i\xi \sigma^\mu (\bar{\theta} + \bar{\rho}) - i(\theta + \rho) \sigma^\mu \bar{\xi} \\
&= G(x^\mu + a^\mu + b^\mu + i\theta \sigma^\mu \bar{\rho} - i\rho \sigma^\mu \bar{\theta} + i\xi \sigma^\mu (\bar{\theta} + \bar{\rho}) - i(\theta + \rho) \sigma^\mu \bar{\xi}, \theta'' + \xi, \bar{\theta}'' + \bar{\xi})
\end{aligned}$$

なので、

$$\begin{aligned}
& \tilde{L}(b^\mu, \rho, \bar{\rho}) \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
& = G(x^\mu + a^\mu + b^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi} + i(\theta + \xi)\sigma^\mu\bar{\rho} - i\rho\sigma^\mu(\bar{\theta} + \bar{\xi}), \theta + \xi + \rho, \bar{\theta} + \bar{\xi} + \bar{\rho}) \\
& \Rightarrow \tilde{L}(b^\mu, \xi, \bar{\xi}) \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
& = G(x^\mu + a^\mu + b^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi} + i(\theta + \xi)\sigma^\mu\bar{\xi} - i\xi\sigma^\mu(\bar{\theta} + \bar{\xi}), \theta + 2\xi, \bar{\theta} + 2\bar{\xi}) \\
& = G(x^\mu + a^\mu + b^\mu, \theta + 2\xi, \bar{\theta} + 2\bar{\xi}) \\
& \tilde{G}(a^\mu, \xi, \bar{\xi}) \tilde{L}(b^\mu, \rho, \bar{\rho}) G(x^\mu, \theta, \bar{\theta}) \\
& = G(x^\mu + a^\mu + b^\mu + i\theta\sigma^\mu\bar{\rho} - i\rho\sigma^\mu\bar{\theta} + i\xi\sigma^\mu(\bar{\theta} + \bar{\rho}) - i(\theta + \rho)\sigma^\mu\bar{\xi}, \theta + \rho + \xi, \bar{\theta} + \bar{\rho} + \bar{\xi}) \\
& \Rightarrow \tilde{G}(x^\mu, \xi, \bar{\xi}) \tilde{L}(y^\mu, \xi, \bar{\xi}) G(a^\mu, \xi, \bar{\xi}) \\
& = G(x^\mu + a^\mu + b^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta} + i\xi\sigma^\mu(\bar{\theta} + \bar{\xi}) - i(\theta + \xi)\sigma^\mu\bar{\xi}, \theta + 2\xi, \bar{\theta} + 2\bar{\xi}) \\
& = G(x^\mu + a^\mu + b^\mu, \theta + 2\xi, \bar{\theta} + 2\bar{\xi})
\end{aligned}$$

以上から、

$$\begin{aligned}
\tilde{L}(b^\mu, \xi, \bar{\xi}) \tilde{G}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) &= \tilde{G}(a^\mu, \xi, \bar{\xi}) \tilde{L}(b^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
&\Rightarrow \tilde{L}(b^\mu, \xi, \bar{\xi}) \tilde{G}(a^\mu, \xi, \bar{\xi}) = \tilde{G}(a^\mu, \xi, \bar{\xi}) \tilde{L}(b^\mu, \xi, \bar{\xi})
\end{aligned}$$

つまり、

共変微分演算子と超対称変換演算子と可換

である。

共変微分の表式は

$$\tilde{L}(x^\mu, \theta, \bar{\theta}) = \exp(ix^\mu \tilde{D}_\mu + \theta^\alpha \tilde{D}_\alpha + \bar{\theta}_\alpha \tilde{\bar{D}}^\alpha) \equiv \exp(x^\mu D_\mu + \theta^\alpha D_\alpha + \bar{\theta}_\alpha \bar{D}^\alpha)$$

で与えられ、

$$\begin{aligned}
G(x^\mu + a^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) &= \tilde{L}(a^\mu, \xi, \bar{\xi}) G(x^\mu, \theta, \bar{\theta}) \\
\Rightarrow G + (a^\mu + i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}) \frac{\partial G}{\partial x^\mu} + \xi^\alpha \frac{\partial G}{\partial \theta^\alpha} + \bar{\xi}^\alpha \frac{\partial G}{\partial \bar{\theta}^\alpha} &= (1 + a^\mu D_\mu + \xi^\alpha D_\alpha + \bar{\xi}_\alpha \bar{D}^\alpha) G(a^\mu, \xi, \bar{\xi}) \\
\rightarrow D_\mu &= \frac{\partial}{\partial x^\mu} \\
\rightarrow D_\alpha &= \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \\
\rightarrow \bar{D}_\alpha &= -\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu} \leftarrow \bar{\xi}_\alpha \frac{\partial G}{\partial \bar{\theta}^\alpha} = \frac{\partial G}{\partial \bar{\theta}^\alpha} \bar{\xi}^\alpha, \bar{\xi}_\alpha \bar{D}^\alpha = \bar{D}_\alpha \bar{\xi}^\alpha
\end{aligned}$$

になる。以上から、微分形式での超変換演算子と共変微分演算子は、

$$\boxed{
\begin{aligned}
\tilde{S}_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{\tilde{S}}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}, \quad i\tilde{P}_\mu = \frac{\partial}{\partial x^\mu} \\
D_\alpha &= \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\theta \sigma^\mu)_\alpha \frac{\partial}{\partial x^\mu}, \quad D_\mu = \frac{\partial}{\partial x^\mu}
\end{aligned}
}$$

とわかった。