

第3章 超対称性変換#2

Free Lagrangian (Vector)

$$\mathcal{L} = -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{i}{2}(\lambda\sigma^\mu D_\mu\bar{\lambda} + \bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda)$$

$$\delta V^\mu = -i(\varepsilon\sigma^\mu\bar{\lambda} + \bar{\varepsilon}\bar{\sigma}^\mu\lambda), \quad \delta\lambda_\alpha = -\frac{i}{2}(\sigma^{\mu\nu}\varepsilon)_\alpha V_{\mu\nu}, \quad \delta\bar{\lambda}_\alpha = \frac{i}{2}(\bar{\varepsilon}\bar{\sigma}^{\mu\nu})_\alpha V_{\mu\nu}$$

$$D_\mu\lambda = \partial_\mu\lambda + ig[V_\mu, \lambda], \quad D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} + ig[V_\mu, \bar{\lambda}]$$

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + ig[V_\mu, V_\nu]$$

ここで、 ε 変換のみ考える。

$$\delta_\varepsilon\mathcal{L} = [\varepsilon Q, \mathcal{L}]$$

$$\text{with } \delta_\varepsilon V^\mu = -i\varepsilon\sigma^\mu\bar{\lambda}, \quad \delta_\varepsilon\lambda^\alpha = \frac{i}{2}(\varepsilon\sigma^{\mu\nu})^\alpha V_{\mu\nu}$$

$$\text{and } \delta_\varepsilon\bar{\lambda}^\alpha = \delta_\varepsilon\lambda_\alpha = 0$$

まず、個々の場の変換性を調べておく。

$$\delta_\varepsilon V_{\mu\nu} = \partial_\mu(-i\varepsilon\sigma_\nu\bar{\lambda}) - \partial_\nu(-i\varepsilon\sigma_\mu\bar{\lambda}) + ig[-i\varepsilon\sigma_\mu\bar{\lambda}, V_\nu] + ig[V_\mu, -i\varepsilon\sigma_\nu\bar{\lambda}]$$

$$= \partial_\mu(-i\varepsilon\sigma_\nu\bar{\lambda}) + ig[V_\mu, -i\varepsilon\sigma_\nu\bar{\lambda}] - [\partial_\nu(-i\varepsilon\sigma_\mu\bar{\lambda}) + ig[V_\nu, -i\varepsilon\sigma_\mu\bar{\lambda}]]$$

$$= -i(\partial_\mu(\varepsilon\sigma_\nu\bar{\lambda}) + ig[V_\mu, \varepsilon\sigma_\nu\bar{\lambda}]) + i(\partial_\nu(\varepsilon\sigma_\mu\bar{\lambda}) + ig[V_\nu, \varepsilon\sigma_\mu\bar{\lambda}])$$

$$= -iD_\mu(\varepsilon\sigma_\nu\bar{\lambda}) + iD_\nu(\varepsilon\sigma_\mu\bar{\lambda}) = -i\varepsilon\sigma_\nu D_\mu\bar{\lambda} + i\varepsilon\sigma_\mu D_\nu\bar{\lambda}$$

$$\delta_\varepsilon(D_\mu\bar{\lambda}^\alpha) = \delta_\varepsilon(\partial_\mu\bar{\lambda} + ig[V_\mu, \bar{\lambda}]) = ig[\delta_\varepsilon V_\mu, \bar{\lambda}] = ig[-i\varepsilon\sigma^\mu\bar{\lambda}, \bar{\lambda}] = g[\varepsilon\sigma^\mu\bar{\lambda}, \bar{\lambda}]$$

$$\delta_\varepsilon(D_\mu\lambda_\alpha) = \delta_\varepsilon(\partial_\mu\lambda_\alpha + ig[V_\mu, \lambda_\alpha]) = D_\mu\left(-\frac{i}{2}(\sigma^{\rho\tau}\varepsilon)_\alpha V_{\rho\tau}\right) + ig[-i\varepsilon\sigma^\mu\bar{\lambda}, \lambda_\alpha]$$

$$= -\frac{i}{2}(\sigma^{\rho\tau}\varepsilon)_\alpha D_\mu V_{\rho\tau} + g[\varepsilon\sigma^\mu\bar{\lambda}, \lambda_\alpha]$$

これらを用いて、

$$\delta_\varepsilon\mathcal{L} = -\frac{1}{2}(\delta_\varepsilon V_{\mu\nu})V^{\mu\nu} + \frac{i}{2}((\delta_\varepsilon\lambda)\sigma^\mu D_\mu\bar{\lambda} + \lambda\sigma^\mu\delta_\varepsilon(D_\mu\bar{\lambda}) + \bar{\lambda}\bar{\sigma}^\mu\delta_\varepsilon(D_\mu\lambda))$$

$$= -\frac{1}{2}(-i\varepsilon\sigma_\nu D_\mu\bar{\lambda} + i\varepsilon\sigma_\mu D_\nu\bar{\lambda})V^{\mu\nu} + \frac{i}{2}\left(\frac{i}{2}(\varepsilon\sigma^{\rho\tau})^\alpha V_{\rho\tau}\right)(\sigma^\mu D_\mu\bar{\lambda})_\alpha$$

$$+ \frac{i}{2}\lambda\sigma^\mu(g[\varepsilon\sigma^\mu\bar{\lambda}, \bar{\lambda}]) + \frac{i}{2}(\bar{\lambda}\bar{\sigma}^\mu)^\alpha\left(-\frac{i}{2}(\sigma^{\rho\tau}\varepsilon)_\alpha D_\mu V_{\rho\tau} + g[\varepsilon\sigma^\mu\bar{\lambda}, \lambda_\alpha]\right)$$

$$\Leftarrow \frac{i}{2}\lambda\sigma^\mu(g[\varepsilon\sigma^\mu\bar{\lambda}, \bar{\lambda}]) + \frac{i}{2}\bar{\lambda}\bar{\sigma}^\mu g[\varepsilon\sigma^\mu\bar{\lambda}, \lambda_\alpha] = \frac{i}{2}g[\varepsilon\sigma^\mu\bar{\lambda}, \lambda\sigma^\mu\bar{\lambda}] + \frac{i}{2}g[\varepsilon\sigma^\mu\bar{\lambda}, \bar{\lambda}\bar{\sigma}^\mu\lambda]$$

$$= \frac{i}{2}g[\varepsilon\sigma^\mu\bar{\lambda}, \lambda\sigma^\mu\bar{\lambda} + \bar{\lambda}\bar{\sigma}^\mu\lambda] = 0$$

$$= -\frac{1}{2}(-i\varepsilon\sigma_\nu D_\mu\bar{\lambda} + i\varepsilon\sigma_\mu D_\nu\bar{\lambda})V^{\mu\nu} - \frac{1}{4}V_{\rho\tau}\varepsilon\sigma^{\rho\tau}\sigma^\mu D_\mu\bar{\lambda} + \frac{1}{4}\bar{\lambda}\bar{\sigma}^\mu\sigma^{\rho\tau}\varepsilon D_\mu V_{\rho\tau}$$

$$\begin{aligned}
& \Leftarrow \sigma^{\rho\tau} \sigma^\mu = \frac{1}{i} \left(\eta^{\tau\mu} \sigma^\rho - \eta^{\rho\mu} \sigma^\tau - i \varepsilon^{\rho\tau\mu\nu} \sigma_\nu \right), \bar{\sigma}^\mu \sigma^{\rho\tau} = \frac{1}{i} \left(\eta^{\mu\rho} \bar{\sigma}^\tau - \eta^{\mu\tau} \bar{\sigma}^\rho + i \varepsilon^{\mu\rho\tau\nu} \bar{\sigma}_\nu \right) \\
& = -\frac{1}{2} \left(-i \varepsilon_\nu D_\mu \bar{\lambda} + i \varepsilon_\mu D_\nu \bar{\lambda} \right) V^{\mu\nu} - \frac{1}{4} V_{\rho\tau} \varepsilon \frac{1}{i} \left(\eta^{\tau\mu} \sigma^\rho - \eta^{\rho\mu} \sigma^\tau - i \varepsilon^{\rho\tau\mu\nu} \sigma_\nu \right) D_\mu \bar{\lambda} \\
& \quad + \frac{1}{4} \bar{\lambda} \frac{1}{i} \left(\eta^{\mu\rho} \bar{\sigma}^\tau - \eta^{\mu\tau} \bar{\sigma}^\rho + i \varepsilon^{\mu\rho\tau\nu} \bar{\sigma}_\nu \right) \varepsilon D_\mu V_{\rho\tau} \\
& = -\frac{1}{2} \left(-i \varepsilon_\nu D_\mu \bar{\lambda} + i \varepsilon_\mu D_\nu \bar{\lambda} \right) V^{\mu\nu} + \frac{1}{4} \left(V_{\rho\tau} \eta^{\tau\mu} \varepsilon \sigma^\rho D_\mu \bar{\lambda} - V_{\rho\tau} \eta^{\rho\mu} \varepsilon \sigma^\tau D_\mu \bar{\lambda} - V_{\rho\tau} i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu D_\mu \bar{\lambda} \right) \\
& \quad - i \frac{1}{4} \left(\eta^{\mu\rho} \bar{\lambda} \bar{\sigma}^\tau \varepsilon D_\mu V_{\rho\tau} - \eta^{\mu\tau} \bar{\lambda} \bar{\sigma}^\rho \varepsilon D_\mu V_{\rho\tau} + i \varepsilon^{\mu\rho\tau\nu} \bar{\lambda} \bar{\sigma}_\nu \varepsilon D_\mu V_{\rho\tau} \right) \\
& = -\frac{1}{2} \left(-i \varepsilon_\nu D_\mu \bar{\lambda} + i \varepsilon_\mu D_\nu \bar{\lambda} \right) V^{\mu\nu} + i \frac{1}{4} \left(V_{\rho\mu} \varepsilon \sigma^\rho D^\mu \bar{\lambda} - V_{\mu\tau} \varepsilon \sigma^\tau D^\mu \bar{\lambda} - V_{\rho\tau} i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu D_\mu \bar{\lambda} \right) \\
& \quad - i \frac{1}{4} \left(\bar{\lambda} \bar{\sigma}^\tau \varepsilon D^\mu V_{\mu\tau} - \bar{\lambda} \bar{\sigma}^\rho \varepsilon D^\mu V_{\rho\mu} + i \varepsilon^{\mu\rho\tau\nu} \bar{\lambda} \bar{\sigma}_\nu \varepsilon D_\mu V_{\rho\tau} \right) \\
& = -\frac{1}{2} \left(-i \varepsilon_\nu D_\mu \bar{\lambda} + i \varepsilon_\mu D_\nu \bar{\lambda} \right) V^{\mu\nu} + i \frac{1}{4} \left(2 V_{\rho\mu} \varepsilon \sigma^\rho D^\mu \bar{\lambda} - V_{\rho\tau} i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu D_\mu \bar{\lambda} \right) \\
& \quad - i \frac{1}{4} \left(-2 \bar{\lambda} \bar{\sigma}^\rho \varepsilon D^\mu V_{\rho\mu} - i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \bar{\lambda} D_\mu V_{\rho\tau} \right) \\
& = -\frac{1}{2} \left(2 i \varepsilon_\mu D_\nu \bar{\lambda} \right) V^{\mu\nu} \\
& \quad + i \frac{1}{4} \left(2 V_{\mu\nu} \varepsilon \sigma^\mu D^\nu \bar{\lambda} - 2 \varepsilon \sigma^\mu \bar{\lambda} D^\nu V_{\mu\nu} - V_{\rho\tau} i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu D_\mu \bar{\lambda} + i \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \bar{\lambda} D_\mu V_{\rho\tau} \right) \\
& = \frac{1}{2} \left(-2 i \varepsilon_\mu D_\nu \bar{\lambda} + i \varepsilon_\mu D_\nu \bar{\lambda} - i \varepsilon_\mu \bar{\lambda} D_\nu \right) V^{\mu\nu} + \frac{1}{4} \left(\varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) V_{\rho\tau} - \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \bar{\lambda} D_\mu V_{\rho\tau} \right) \\
& = \frac{1}{2} \left(-i \varepsilon_\mu D_\nu \bar{\lambda} - i \varepsilon_\mu \bar{\lambda} D_\nu \right) V^{\mu\nu} + \frac{1}{4} \left(\varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) - \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \bar{\lambda} D_\mu \right) V_{\rho\tau} \\
& = -\frac{1}{2} i D_\nu \left(\varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} \right) - \frac{1}{4} \varepsilon^{\rho\tau\mu\nu} D_\nu \left(\varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right) + \frac{1}{2} \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) V_{\rho\tau} \\
& \Leftarrow D_\nu \left(\varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} \right) = \partial_\nu \left(\varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} \right) \\
& = \partial_\nu \left(-\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} - \frac{1}{4} \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right) + \frac{1}{2} \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) V_{\rho\tau}
\end{aligned}$$

ここで、次の関係式を使用する：

$$\begin{aligned}
\delta_\varepsilon \left(\varepsilon^{\mu\nu\rho\tau} V_{\mu\nu} V_{\rho\tau} \right) &= \varepsilon^{\mu\nu\rho\tau} \left[\left(\delta_\varepsilon V_{\mu\nu} \right) V_{\rho\tau} + V_{\mu\nu} \left(\delta_\varepsilon V_{\rho\tau} \right) \right] \\
&\Leftarrow \delta_\varepsilon V_{\mu\nu} = -i \varepsilon \sigma_\nu D_\mu \bar{\lambda} + i \varepsilon \sigma_\mu D_\nu \bar{\lambda} \\
&= \varepsilon^{\mu\nu\rho\tau} \left[\left(-i \varepsilon \sigma_\nu D_\mu \bar{\lambda} + i \varepsilon \sigma_\mu D_\nu \bar{\lambda} \right) V_{\rho\tau} + V_{\mu\nu} \left(-i \varepsilon \sigma_\rho D_\tau \bar{\lambda} + i \varepsilon \sigma_\tau D_\rho \bar{\lambda} \right) \right] \\
&= \varepsilon^{\mu\nu\rho\tau} \left(-4 i \varepsilon \sigma_\nu D_\mu \bar{\lambda} \right) V_{\rho\tau} = -4 i \varepsilon^{\mu\nu\rho\tau} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) V_{\rho\tau} \\
&\Leftarrow \tilde{V}^{\mu\nu} = \frac{1}{2!} \varepsilon^{\mu\nu\rho\tau} V_{\rho\tau} \\
\delta_\varepsilon \left(\tilde{V}^{\mu\nu} V_{\rho\tau} \right) &= -2 i \varepsilon^{\mu\nu\rho\tau} \varepsilon \sigma_\nu \left(D_\mu \bar{\lambda} \right) V_{\rho\tau}
\end{aligned}$$

そこで、 $\delta_\varepsilon \mathcal{L}$ は、

$$\begin{aligned}
\delta_\varepsilon \mathcal{L} &= \partial_\nu \left(-\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} - \frac{1}{4} \varepsilon^{\rho\tau\nu\mu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right) + \frac{1}{2} \varepsilon^{\rho\tau\mu\nu} \varepsilon \sigma_\nu (D_\mu \bar{\lambda}) V_{\rho\tau} \\
&\Leftarrow \delta_\varepsilon (\tilde{V}^{\mu\nu} V_{\rho\tau}) = -2i \varepsilon^{\mu\nu\rho\tau} \varepsilon \sigma_\nu (D_\mu \bar{\lambda}) V_{\rho\tau} \\
&= \partial_\nu \left(-\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\mu\nu} - \frac{1}{4} \varepsilon^{\rho\tau\nu\mu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right) + \frac{1}{2} \left(-\frac{1}{2i} \delta_\varepsilon (\tilde{V}^{\mu\nu} V_{\rho\tau}) \right) \\
&= \partial_\nu \left(\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\nu\mu} - \frac{1}{4} \varepsilon^{\rho\tau\nu\mu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right) + \delta_\varepsilon \left(\frac{i}{4} \tilde{V}^{\mu\nu} V_{\rho\tau} \right)
\end{aligned}$$

故に、

$$\delta_\varepsilon \left(\mathcal{L} - \frac{i}{4} \tilde{V}^{\mu\nu} V_{\rho\tau} \right) = \partial_\nu \left(\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\nu\mu} - \frac{1}{4} \varepsilon^{\rho\tau\nu\mu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right)$$

を得る。以上から、

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{i}{2} (\lambda \sigma^\mu D_\mu \bar{\lambda} + \bar{\lambda} \bar{\sigma}^\mu D_\mu \lambda) \\
\delta V^\mu &= -i (\varepsilon \sigma^\mu \bar{\lambda} + \bar{\varepsilon} \bar{\sigma}^\mu \lambda), \quad \delta \lambda_\alpha = -\frac{i}{2} (\sigma^{\mu\nu} \varepsilon)_\alpha V_{\mu\nu}, \quad \delta \bar{\lambda}_\alpha = \frac{i}{2} (\bar{\varepsilon} \bar{\sigma}^{\mu\nu})_\alpha V_{\mu\nu} \\
D_\mu \lambda &= \partial_\mu \lambda + ig [V_\mu, \lambda], \quad D_\mu \bar{\lambda} = \partial_\mu \bar{\lambda} + ig [V_\mu, \bar{\lambda}] \\
V_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + ig [V_\mu, V_\nu] \\
\delta_\varepsilon \left(\mathcal{L} - \frac{i}{4} \tilde{V}^{\mu\nu} V_{\rho\tau} \right) &= \partial_\nu \left(\frac{1}{2} i \varepsilon \sigma_\mu \bar{\lambda} V^{\nu\mu} - \frac{1}{4} \varepsilon^{\rho\tau\nu\mu} \varepsilon \sigma_\mu \bar{\lambda} V_{\rho\tau} \right)
\end{aligned}$$