

## 第2章 超対称性変換

### Free Lagrangian (Chiral)

$$\mathcal{L} = |\partial_\mu \phi|^2 + i\psi\sigma^\mu \partial_\mu \bar{\psi}$$

$$\begin{aligned} \delta\phi &= \sqrt{2}\varepsilon\psi, & \delta\psi_\alpha &= -a\sqrt{2}i(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi, & \delta\psi^\alpha &= a\sqrt{2}i(\bar{\varepsilon}\sigma^\mu)^\alpha \partial_\mu \phi \\ \delta\bar{\psi}_\alpha &= a\sqrt{2}i(\varepsilon\sigma^\mu)_\alpha \partial_\mu \phi^*, & \delta\bar{\psi}^\alpha &= -a\sqrt{2}i(\bar{\sigma}^\mu \varepsilon)^\alpha \partial_\mu \phi^* \end{aligned} \quad (a\text{を決める。})$$

$$\mathcal{L} = |\partial_\mu \phi|^2 + \frac{i}{2}(\psi\sigma^\mu \partial_\mu \bar{\psi} + \bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi)$$

$$\delta\mathcal{L} = \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}) \partial_\mu \phi + \partial^\mu \phi^\dagger \partial_\mu (\sqrt{2}\varepsilon\psi)$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\bar{\varepsilon}\bar{\sigma}^\nu)^\alpha \partial_\nu \phi (\sigma^\mu \partial_\mu \bar{\psi})_\alpha + (\psi\sigma^\mu)_\alpha \partial_\mu (-a\sqrt{2}i(\bar{\sigma}^\nu \varepsilon)^\alpha \partial_\nu \phi^\dagger) \right]$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\varepsilon\sigma^\nu)_\alpha \partial_\nu \phi^\dagger (\bar{\sigma}^\mu \partial_\mu \psi)^\alpha + (\bar{\psi}\bar{\sigma}^\mu)^\alpha \partial_\mu (-a\sqrt{2}i(\sigma^\nu \bar{\varepsilon})_\alpha \partial_\nu \phi) \right]$$

$$= \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}) \partial_\mu \phi + \partial^\mu \phi^\dagger \partial_\mu (\sqrt{2}\varepsilon\psi)$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\bar{\varepsilon}\bar{\sigma}^\nu \sigma^\mu \partial_\mu \bar{\psi}) \partial_\nu \phi - a\sqrt{2}i(\psi\sigma^\mu \bar{\sigma}^\nu \varepsilon) (\partial_\mu \partial_\nu \phi^\dagger) \right]$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\varepsilon\sigma^\nu \bar{\sigma}^\mu \partial_\mu \psi) \partial_\nu \phi^\dagger - a\sqrt{2}i(\bar{\psi}\bar{\sigma}^\mu \sigma^\nu \bar{\varepsilon}) (\partial_\mu \partial_\nu \phi) \right] \leftarrow \bar{\sigma}^\nu \sigma^\mu = g^{\nu\mu} + i\bar{\sigma}^{\nu\mu}, \quad \sigma^\nu \bar{\sigma}^\mu = g^{\nu\mu} + i\sigma^{\nu\mu}$$

$$= \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}) \partial_\mu \phi + \partial^\mu \phi^\dagger \partial_\mu (\sqrt{2}\varepsilon\psi)$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\bar{\varepsilon}\bar{\sigma}^\mu \bar{\psi}) \partial_\mu \phi + a\sqrt{2}i(\bar{\varepsilon}i\bar{\sigma}^{\nu\mu} \partial_\mu \bar{\psi}) \partial_\nu \phi - a\sqrt{2}i(\psi\varepsilon)(\square\phi^\dagger) - a\sqrt{2}i(\psi i\sigma^{\mu\nu} \varepsilon)(\partial_\mu \partial_\nu \phi^\dagger) \right]$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\varepsilon\partial^\mu \psi) \partial_\mu \phi^\dagger + a\sqrt{2}i(\varepsilon i\sigma^{\nu\mu} \partial_\mu \psi) \partial_\nu \phi^\dagger - a\sqrt{2}i(\bar{\psi}\bar{\varepsilon})(\square\phi) - a\sqrt{2}i(\bar{\psi}i\bar{\sigma}^{\mu\nu} \bar{\varepsilon})(\partial_\mu \partial_\nu \phi) \right]$$

$$= \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}) \partial_\mu \phi + \partial^\mu \phi^\dagger \partial_\mu (\sqrt{2}\varepsilon\psi)$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\bar{\varepsilon}\bar{\sigma}^\mu \bar{\psi}) \partial_\mu \phi - a\sqrt{2}(\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \partial_\mu \bar{\psi}) \partial_\nu \phi - a\sqrt{2}i(\psi\varepsilon)(\square\phi^\dagger) \right]$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i(\varepsilon\partial^\mu \psi) \partial_\mu \phi^\dagger - a\sqrt{2}(\varepsilon\sigma^{\nu\mu} \partial_\mu \psi) \partial_\nu \phi^\dagger - a\sqrt{2}i(\bar{\psi}\bar{\varepsilon})(\square\phi) \right]$$

$$\leftarrow \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}) \partial_\mu \phi = \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}\partial_\mu \phi) - (\sqrt{2}\bar{\varepsilon}\bar{\psi})\square\phi$$

$$= \partial^\mu (\sqrt{2}\bar{\varepsilon}\bar{\psi}\partial_\mu \phi) - (\sqrt{2}\bar{\varepsilon}\bar{\psi})\square\phi + \partial^\mu (\partial_\mu \phi^\dagger \sqrt{2}\varepsilon\psi) - \square\phi^\dagger (\sqrt{2}\varepsilon\psi)$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i\partial^\mu (\bar{\varepsilon}\bar{\psi}\partial_\mu \phi) - a\sqrt{2}i(\bar{\varepsilon}\bar{\psi}\square\phi) - a\sqrt{2}i(\psi\varepsilon)(\square\phi^\dagger) \right]$$

$$+ \frac{i}{2} \left[ a\sqrt{2}i\partial^\mu (\varepsilon\psi\partial_\mu \phi^\dagger) - a\sqrt{2}i(\varepsilon\psi)\square\phi^\dagger - a\sqrt{2}i(\bar{\psi}\bar{\varepsilon})(\square\phi) \right]$$

$$- \frac{i}{2} \left[ a\sqrt{2}(\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \partial_\mu \bar{\psi}) \partial_\nu \phi + a\sqrt{2}(\varepsilon\sigma^{\nu\mu} \partial_\mu \psi) \partial_\nu \phi^\dagger \right]$$

$$\leftarrow (\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \partial_\mu \bar{\psi}) \partial_\nu \phi = \partial_\mu (\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \bar{\psi}\partial_\nu \phi) - (\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \bar{\psi}) \partial_\mu \partial_\nu \phi = \partial_\mu (\bar{\varepsilon}\bar{\sigma}^{\nu\mu} \bar{\psi}\partial_\nu \phi)$$

$$\begin{aligned}
&= \partial^\mu \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \partial_\mu \phi \right) - \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \right) \square \phi + \partial^\mu \left( \partial_\mu \phi^\dagger \sqrt{2} \varepsilon \psi \right) - \square \phi^\dagger \left( \sqrt{2} \varepsilon \psi \right) \\
&+ \frac{i}{2} \left[ a \sqrt{2} i \partial^\mu \left( \bar{\varepsilon} \bar{\psi} \partial_\mu \phi \right) - a \sqrt{2} i \left( \bar{\varepsilon} \bar{\psi} \square \phi \right) - a \sqrt{2} i \left( \psi \varepsilon \right) \left( \square \phi^\dagger \right) \right] \\
&+ \frac{i}{2} \left[ a \sqrt{2} i \partial^\mu \left( \varepsilon \psi \partial_\mu \phi^\dagger \right) - a \sqrt{2} i \left( \varepsilon \psi \square \phi^\dagger \right) - a \sqrt{2} i \left( \bar{\psi} \varepsilon \right) \left( \square \phi \right) \right] \\
&- \frac{i}{2} \left[ a \sqrt{2} \partial_\mu \left( \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) + a \sqrt{2} \partial_\mu \left( \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \right] \\
&= \partial^\mu \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \partial_\mu \phi \right) - \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \right) \square \phi + \partial^\mu \left( \partial_\mu \phi^\dagger \sqrt{2} \varepsilon \psi \right) - \square \phi^\dagger \left( \sqrt{2} \varepsilon \psi \right) \\
&+ \frac{i}{2} \left[ a \sqrt{2} i \partial^\mu \left( \bar{\varepsilon} \bar{\psi} \partial_\mu \phi \right) + a \sqrt{2} i \partial^\mu \left( \varepsilon \psi \partial_\mu \phi^\dagger \right) \right] + a \sqrt{2} \left[ \left( \bar{\varepsilon} \bar{\psi} \square \phi \right) + \left( \psi \varepsilon \right) \left( \square \phi^\dagger \right) \right] \\
&- \frac{i}{2} \left[ a \sqrt{2} \partial_\mu \left( \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) + a \sqrt{2} \partial_\mu \left( \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \right] \\
&= \partial_\mu \left( \left( 1 - \frac{a}{2} \right) \sqrt{2} \bar{\varepsilon} \bar{\psi} \partial^\mu \phi - \frac{i}{2} a \sqrt{2} \left( \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) \right) + \partial_\mu \left( \left( 1 - \frac{a}{2} \right) \sqrt{2} \varepsilon \psi \partial^\mu \phi^\dagger - \frac{i}{2} a \sqrt{2} \left( \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \right) \\
&- \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \right) \square \phi (1-a) - \square \phi^\dagger \left( \sqrt{2} \varepsilon \psi \right) (1-a) \\
&= \sqrt{2} \partial_\mu \left( \left( 1 - \frac{a}{2} \right) \bar{\varepsilon} \bar{\psi} \partial^\mu \phi - \frac{i}{2} a \left( \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) \right) + \sqrt{2} \partial_\mu \left( \left( 1 - \frac{a}{2} \right) \varepsilon \psi \partial^\mu \phi^\dagger - \frac{i}{2} a \left( \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \right) \\
&- \left( \sqrt{2} \bar{\varepsilon} \bar{\psi} \right) \square \phi (1-a) - \square \phi^\dagger \left( \sqrt{2} \varepsilon \psi \right) (1-a)
\end{aligned}$$

ここで、

$$a = 1$$

とすれば、

$$\begin{aligned}
&= \sqrt{2} \partial_\mu \left( \frac{1}{2} \bar{\varepsilon} \bar{\psi} \partial_\mu \phi - \frac{i}{2} \left( \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) \right) + \sqrt{2} \partial_\mu \left( \frac{1}{2} \varepsilon \psi \partial_\mu \phi^\dagger - \frac{i}{2} \left( \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \right) \\
&= \frac{\sqrt{2}}{2} \partial_\mu \left( \bar{\varepsilon} \bar{\psi} \partial_\mu \phi - i \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi \right) + \frac{\sqrt{2}}{2} \partial_\mu \left( \varepsilon \psi \partial_\mu \phi^\dagger - i \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \\
&= \frac{\sqrt{2}}{2} \partial_\mu \left( \bar{\varepsilon} \bar{\psi} \partial^\mu \phi - i \bar{\varepsilon} \bar{\sigma}^{\nu\mu} \bar{\psi} \partial_\nu \phi + \varepsilon \psi \partial^\mu \phi^\dagger - i \varepsilon \sigma^{\nu\mu} \psi \partial_\nu \phi^\dagger \right) \\
&\quad \leftarrow \bar{\sigma}^\mu \sigma^\nu = g^{\mu\nu} + i \bar{\sigma}^{\mu\nu} = g^{\mu\nu} - i \bar{\sigma}^{\nu\mu}, \quad \sigma^\mu \bar{\sigma}^\nu = g^{\mu\nu} - i \sigma^{\nu\mu} \\
&= \frac{\sqrt{2}}{2} \partial_\mu \left( \bar{\varepsilon} \bar{\sigma}^\mu \sigma^\nu \bar{\psi} \partial_\nu \phi + \varepsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^\dagger \right)
\end{aligned}$$

以上から、

$$\begin{aligned}
\mathcal{L} &= \left| \partial_\mu \phi \right|^2 + i \psi \sigma^\mu \partial_\mu \bar{\psi} \Rightarrow \delta \mathcal{L} = \frac{\sqrt{2}}{2} \partial_\mu \left( \bar{\varepsilon} \bar{\sigma}^\mu \sigma^\nu \bar{\psi} \partial_\nu \phi + \varepsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^\dagger \right) \\
\delta \phi &= \sqrt{2} \varepsilon \psi \\
\delta \psi_\alpha &= -\sqrt{2} i \left( \sigma^\mu \bar{\varepsilon} \right)_\alpha \partial_\mu \phi, \quad \delta \bar{\psi}^\alpha = \sqrt{2} i \left( \bar{\varepsilon} \sigma^\mu \right)^\alpha \partial_\mu \phi \\
\delta \bar{\psi}_\alpha &= \sqrt{2} i \left( \varepsilon \sigma^\mu \right)_\alpha \partial_\mu \phi^*, \quad \delta \bar{\psi}^\alpha = -\sqrt{2} i \left( \bar{\sigma}^\mu \varepsilon \right)^\alpha \partial_\mu \phi^*
\end{aligned}$$

## 超対称性演算子の交換関係

超対称変換の generators を  $S_\alpha$  と  $\bar{S}_\alpha$  すると、任意の場  $A$  については「場の変換性と演算子」

$$\hat{\phi}(X') = U^{\dagger-1} \hat{\phi}(X) U^{-1} = U \hat{\phi}(X) U^\dagger \quad \text{with } X' = X + A$$

$$U = \exp\left(-i \sum_A a^A Q^A\right) \quad \text{with } \left(\sum_A a^A Q^A\right)^\dagger = \sum_A a^A Q^A$$

$$\delta \hat{\phi}(X) = \left[ i \sum_A a^A Q^A, \hat{\phi}(X) \right] = -A^\alpha \frac{\partial \hat{\phi}(X)}{\partial X^\alpha}$$

のまとめより  $i \sum_A a^A Q^A = \varepsilon S + \bar{\varepsilon} \bar{S}$  with  $(\varepsilon S)^\dagger = \bar{\varepsilon} \bar{S}$  (或いは  $i \sum_A a^A Q^A = i(\varepsilon S - \bar{\varepsilon} \bar{S})$  でもよい) で、

$$\delta A = [\varepsilon S + \bar{\varepsilon} \bar{S}, A]$$

になる。一方、場の変換性を用いて

$$\begin{aligned} [\delta_1, \delta_2] \phi &= \delta_1 \delta_2 \phi - (1 \leftrightarrow 2) = \delta_1 \left( \sqrt{2} \varepsilon_2^\alpha \psi_\alpha \right) - (1 \leftrightarrow 2) = \sqrt{2} \varepsilon_2^\alpha \left( -\sqrt{2} i (\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi \right) - (1 \leftrightarrow 2) \\ &= -2i \varepsilon_2 \sigma^\mu \bar{\varepsilon}_1 \partial_\mu \phi + 2i \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu \phi \Leftarrow \varepsilon_2 \sigma^\mu \bar{\varepsilon}_1 = -\bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 \\ &= 2i \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 \partial_\mu \phi + 2i \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu \phi = 2i \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) \partial_\mu \phi \end{aligned}$$

$$\begin{aligned} [\delta_1, \delta_2] \psi_\alpha &= \delta_1 \delta_2 \psi_\alpha - (1 \leftrightarrow 2) = \delta_1 \left( -\sqrt{2} i (\sigma^\mu \bar{\varepsilon}_2)_\alpha \partial_\mu \phi \right) - (1 \leftrightarrow 2) = -\sqrt{2} i (\sigma^\mu \bar{\varepsilon}_2)_\alpha \partial_\mu \left( \sqrt{2} \varepsilon_1^\beta \psi_\beta \right) - (1 \leftrightarrow 2) \\ &= -2i (\sigma^\mu \bar{\varepsilon}_2)_\alpha \left( \varepsilon_1^\beta \partial_\mu \psi_\beta \right) - (1 \leftrightarrow 2) \end{aligned}$$

ここで、Fierz identity を使用する。Fierz identity は

$$\delta_\alpha^\beta \delta_\delta^\gamma = \frac{1}{2} (\sigma^\mu)_{\alpha\delta} (\bar{\sigma}_\mu)^{\beta\gamma}, \quad \delta_\alpha^\beta \delta_\delta^\gamma = \frac{1}{2} \left( \delta_\alpha^\gamma \delta_\delta^\beta + \frac{1}{4} (\sigma^{\mu\nu})_{\alpha\delta}^\gamma (\sigma_{\mu\nu})_{\delta\alpha}^\beta \right)$$

$(\sigma^\mu \bar{\varepsilon}_2)_\alpha (\varepsilon_1^\beta \partial_\mu \psi_\beta)$  を計算し  $(\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2) \psi$  型に変形する。両辺に  $\theta^\alpha \varphi_\beta \chi^\delta (\sigma^\rho \bar{\eta})_\gamma$  を乗ずる :

$$\begin{aligned} \theta^\alpha \varphi_\beta \chi^\delta (\sigma^\rho \bar{\eta})_\gamma \delta_\alpha^\beta \delta_\delta^\gamma &= \frac{1}{2} \left( \theta^\alpha \varphi_\beta \chi^\delta (\sigma^\rho \bar{\eta})_\gamma \delta_\alpha^\gamma \delta_\delta^\beta + \frac{1}{4} \theta^\alpha \varphi_\beta \chi^\delta (\sigma^\rho \bar{\eta})_\gamma (\sigma^{\mu\nu})_{\alpha\delta}^\gamma (\sigma_{\mu\nu})_{\delta\alpha}^\beta \right) \\ (\theta \varphi) (\chi \sigma^\rho \bar{\eta}) &= \frac{1}{2} \left[ -(\chi \varphi) (\theta \sigma^\rho \bar{\eta}) + \frac{1}{4} (\theta \sigma^{\mu\nu} \sigma^\rho \bar{\eta}) (-\chi \sigma_{\mu\nu} \varphi) \right] = -\frac{1}{2} \left[ (\chi \varphi) (\theta \sigma^\rho \bar{\eta}) + \frac{1}{4} (\theta \sigma^{\mu\nu} \sigma^\rho \bar{\eta}) (\chi \sigma_{\mu\nu} \varphi) \right] \\ &\Leftarrow i \sigma^{\mu\nu} \sigma^\rho = \eta^{\nu\rho} \sigma^\mu - \eta^{\mu\rho} \sigma^\nu - i \varepsilon^{\mu\nu\rho\tau} \sigma_\tau \\ &= -\frac{1}{2} \left[ (\varphi \chi) (\theta \sigma^\rho \bar{\eta}) + \frac{1}{4} \frac{1}{i} (\theta (\eta^{\nu\rho} \sigma^\mu - \eta^{\mu\rho} \sigma^\nu - i \varepsilon^{\mu\nu\rho\tau} \sigma_\tau) \bar{\eta}) (\chi \sigma_{\mu\nu} \varphi) \right] \\ &= -\frac{1}{2} \left[ (\varphi \chi) (\theta \sigma^\rho \bar{\eta}) - \frac{1}{4} i (\theta \sigma_\mu \bar{\eta}) (\chi \sigma^{\mu\rho} \varphi) + \frac{1}{4} i (\theta \sigma_\nu \bar{\eta}) (\chi \sigma^{\rho\nu} \varphi) - \frac{1}{4} \varepsilon^{\mu\nu\rho\tau} (\theta \sigma_\tau \bar{\eta}) (\chi \sigma_{\mu\nu} \varphi) \right] \\ &= -\frac{1}{2} \left[ (\varphi \chi) (\theta \sigma^\rho \bar{\eta}) - \frac{1}{2} i (\theta \sigma_\mu \bar{\eta}) (\chi \sigma^{\mu\rho} \varphi) - \frac{1}{4} \varepsilon^{\mu\nu\rho\tau} (\theta \sigma_\tau \bar{\eta}) (\chi \sigma_{\mu\nu} \varphi) \right] \\ &\Leftarrow \sigma^\mu \bar{\sigma}^\rho = g^{\mu\rho} + i \sigma^{\mu\rho} \rightarrow i \sigma^{\mu\rho} = -g^{\mu\rho} + \sigma^\mu \bar{\sigma}^\rho \\ &= -\frac{1}{2} \left[ (\varphi \chi) (\theta \sigma^\rho \bar{\eta}) - \frac{1}{2} (\theta \sigma_\mu \bar{\eta}) (\chi (-g^{\mu\rho} + \sigma^\mu \bar{\sigma}^\rho) \varphi) - \frac{1}{4} \varepsilon^{\mu\nu\rho\tau} (\theta \sigma_\tau \bar{\eta}) (\chi (-g_{\mu\nu} + \sigma_\mu \bar{\sigma}_\nu) \varphi) \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \left[ (\varphi\chi)(\theta\sigma^\rho\bar{\eta}) + \frac{1}{2}(\theta\sigma^\rho\bar{\eta})(\chi\varphi) - \frac{1}{2}(\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) + \frac{1}{4}i\varepsilon^{\mu\nu\rho\tau}(\theta\sigma_\tau\bar{\eta})(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \right] \\
&= -\frac{1}{4} \left[ 3(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) + \frac{1}{2}i\varepsilon^{\mu\nu\rho\tau}(\theta\sigma_\tau\bar{\eta})(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \right] \\
&= -\frac{1}{4} \left[ 3(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) - \frac{1}{2}i\varepsilon^{\mu\nu\rho\tau}(\bar{\eta}\bar{\sigma}_\tau\theta)(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \right] \\
&\Leftarrow 2i\varepsilon^{\mu\nu\rho\tau}\bar{\sigma}_\tau = \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu, \quad (\theta\sigma_\tau\bar{\eta}) = -(\bar{\eta}\bar{\sigma}_\tau\theta) \\
&= -\frac{1}{4} \left[ 3(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) - \frac{1}{2} \frac{1}{2}(\bar{\eta}(\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho - \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu)\theta)(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \right] \\
&(*) (\bar{\eta}\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\theta)(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \Leftarrow (\sigma^\mu)_{\alpha\delta}(\bar{\sigma}_\mu)^{\gamma\beta} = 2\delta_\alpha^\beta\delta_\delta^\gamma \\
&= \bar{\eta}_\gamma(\sigma^\nu\bar{\sigma}^\rho\theta)_\beta(\bar{\sigma}^\mu)^{\gamma\beta}(\sigma_\mu)_{\alpha\delta}\chi^\alpha(\bar{\sigma}_\nu\varphi)^\delta = \bar{\eta}_\gamma(\sigma^\nu\bar{\sigma}^\rho\theta)_\beta 2\delta_\alpha^\beta\delta_\delta^\gamma\chi^\alpha(\bar{\sigma}_\nu\varphi)^\delta \\
&= 2(\sigma^\nu\bar{\sigma}^\rho\theta)_\beta\delta_\alpha^\beta\chi^\alpha(\bar{\eta}\bar{\sigma}_\nu\varphi) = -2(\chi\sigma^\nu\bar{\sigma}^\rho\theta)(\bar{\eta}\bar{\sigma}_\nu\varphi) \\
&= -2\chi^\alpha(\bar{\sigma}^\rho\theta)^\delta\bar{\eta}_\gamma\varphi_\beta(\sigma^\nu)_{\alpha\delta}(\bar{\sigma}_\nu)^{\gamma\beta} = -2\chi^\alpha(\bar{\sigma}^\rho\theta)^\delta\bar{\eta}_\gamma\varphi_\beta 2\delta_\alpha^\beta\delta_\delta^\gamma \\
&= 4(\chi\varphi)(\bar{\eta}\bar{\sigma}^\rho\theta) = -4(\chi\varphi)(\theta\sigma^\rho\bar{\eta}) \\
&(*) (\bar{\eta}\bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu\theta)(\chi\sigma_\mu\bar{\sigma}_\nu\varphi) \Leftarrow (\sigma^\mu)_{\alpha\delta}(\bar{\sigma}_\mu)^{\gamma\beta} = 2\delta_\alpha^\beta\delta_\delta^\gamma \\
&= (\bar{\eta}\bar{\sigma}^\rho\sigma^\nu)_\gamma\theta_\beta(\bar{\sigma}^\mu)^{\gamma\beta}(\sigma_\mu)_{\alpha\delta}\chi^\alpha(\bar{\sigma}_\nu\varphi)^\delta = (\bar{\eta}\bar{\sigma}^\rho\sigma^\nu)_\gamma\theta_\beta 2\delta_\alpha^\beta\delta_\delta^\gamma\chi^\alpha(\bar{\sigma}_\nu\varphi)^\delta \\
&= -2(\bar{\eta}\bar{\sigma}^\rho\sigma^\nu\bar{\sigma}_\nu\varphi)(\chi\theta) \Leftarrow (\sigma^\mu\bar{\sigma}_\mu)_\alpha^\beta = 4\delta_\alpha^\beta \\
&= -8(\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta) \\
&= -\frac{1}{4} \left[ 3(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) - \frac{1}{2} \frac{1}{2}(-4(\chi\varphi)(\theta\sigma^\rho\bar{\eta}) + 8(\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta)) \right] \\
&= -\frac{1}{4} \left[ 3(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) + (\chi\varphi)(\theta\sigma^\rho\bar{\eta}) - 2(\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta) \right] \\
&= -\frac{1}{4} \left[ 4(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - (\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) - 2(\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta) \right] \\
&= -\frac{1}{2} \left[ 2(\varphi\chi)(\theta\sigma^\rho\bar{\eta}) - \frac{1}{2}(\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi) - (\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta) \right]
\end{aligned}$$

ここで、

$$\begin{aligned}
\theta\sigma_\mu\bar{\eta} \cdot \varphi\sigma^\rho\bar{\sigma}^\mu\chi &= \theta^\alpha(\sigma_\mu)_{\alpha\delta}\bar{\eta}^\delta \cdot (\varphi\sigma^\rho)_\gamma(\bar{\sigma}^\mu)^{\gamma\beta}\chi_\beta \Leftarrow (\sigma^\mu)_{\alpha\delta}(\bar{\sigma}_\mu)^{\gamma\beta} = 2\delta_\alpha^\beta\delta_\delta^\gamma \\
&= 2\delta_\alpha^\beta\delta_\delta^\gamma\theta^\alpha\bar{\eta}^\delta \cdot (\varphi\sigma^\rho)_\gamma\chi_\beta = 2\delta_\delta^\gamma\theta\chi \cdot \bar{\eta}^\delta(\varphi\sigma^\rho)_\gamma = -2\theta\chi \cdot \varphi\sigma^\rho\bar{\eta} \\
\theta\sigma_\mu\bar{\eta} \cdot \varphi\sigma^\mu\bar{\sigma}^\rho\chi &= \theta\sigma_\mu\bar{\eta} \cdot \chi\sigma^\rho\bar{\sigma}^\mu\varphi = -2\theta\varphi \cdot \chi\sigma^\rho\bar{\eta} \\
\theta\sigma_\mu\bar{\eta} \cdot \varphi\sigma^{\rho\mu}\chi &= \frac{1}{2i}\theta\sigma_\mu\bar{\eta} \cdot \varphi(\sigma^\rho\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^\rho)\chi = \frac{1}{2i}\theta\sigma_\mu\bar{\eta} \cdot \varphi\sigma^\rho\bar{\sigma}^\mu\chi - \frac{1}{2i}\theta\sigma_\mu\bar{\eta} \cdot \varphi\bar{\sigma}^\mu\sigma^\rho\chi \\
&= \frac{1}{2i}(-2\theta\chi \cdot \varphi\sigma^\rho\bar{\eta}) - \frac{1}{2i}(-2\theta\varphi \cdot \chi\sigma^\rho\bar{\eta}) = (i\theta\chi \cdot \varphi\sigma^\rho\bar{\eta}) - (i\theta\varphi \cdot \chi\sigma^\rho\bar{\eta}) \\
&= i(\theta\chi \cdot \varphi\sigma^\rho\bar{\eta} - \theta\varphi \cdot \chi\sigma^\rho\bar{\eta})
\end{aligned}$$

に注意すると、

$$\begin{aligned}\theta\sigma_\mu\bar{\eta}\cdot\varphi\sigma^\rho\bar{\sigma}^\mu\chi &= -2\theta\chi\cdot\varphi\sigma^\rho\bar{\eta} \\ \theta\sigma_\mu\bar{\eta}\cdot\varphi\sigma^\mu\bar{\sigma}^\rho\chi &= -2\theta\varphi\cdot\chi\sigma^\rho\bar{\eta} \\ \theta\sigma_\mu\bar{\eta}\cdot\varphi\sigma^{\rho\mu}\chi &= i(\theta\chi\cdot\varphi\sigma^\rho\bar{\eta}-\theta\varphi\cdot\chi\sigma^\rho\bar{\eta})\end{aligned}$$

がわかる。これより

$$\begin{aligned}(\theta\varphi)(\chi\sigma^\rho\bar{\eta}) &= -\frac{1}{2}\left[2(\varphi\chi)(\theta\sigma^\rho\bar{\eta})-\frac{1}{2}(\theta\sigma_\mu\bar{\eta})(\chi\sigma^\mu\bar{\sigma}^\rho\varphi)-(\bar{\eta}\bar{\sigma}^\rho\varphi)(\chi\theta)\right] \\ &\Leftarrow \theta\sigma_\mu\bar{\eta}\cdot\varphi\sigma^\mu\bar{\sigma}^\rho\chi = -2\theta\varphi\cdot\chi\sigma^\rho\bar{\eta} \text{ with } \varphi \leftrightarrow \chi \\ &= -\frac{1}{2}\left[2(\varphi\chi)(\theta\sigma^\rho\bar{\eta})-\frac{1}{2}(-2\theta\chi\cdot\varphi\sigma^\rho\bar{\eta})+(\varphi\sigma^\rho\bar{\eta})(\chi\theta)\right] \\ &= -\frac{1}{2}\left[2(\varphi\chi)(\theta\sigma^\rho\bar{\eta})+2(\varphi\sigma^\rho\bar{\eta})(\chi\theta)\right] = -\left[(\varphi\chi)(\theta\sigma^\rho\bar{\eta})+(\chi\theta)(\varphi\sigma^\rho\bar{\eta})\right]\end{aligned}$$

そこで

$$(\theta\varphi)(\chi\sigma^\rho\bar{\eta}) = -\left[(\varphi\chi)(\theta\sigma^\rho\bar{\eta})+(\chi\theta)(\varphi\sigma^\rho\bar{\eta})\right] \Rightarrow (\theta\varphi)(\sigma^\rho\bar{\eta})_\alpha = -\left[\varphi_\alpha(\theta\sigma^\rho\bar{\eta})+\theta_\alpha(\varphi\sigma^\rho\bar{\eta})\right]$$

に

$$\bar{\eta} = \bar{\varepsilon}_2, \quad \theta = \varepsilon_1, \quad \varphi = \partial_\mu\psi$$

を代入して、

$$\begin{aligned}(\varepsilon_1\partial_\rho\psi)(\sigma^\rho\bar{\varepsilon}_2)_\alpha &= -\left[\partial_\rho\psi_\alpha(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2)+\varepsilon_{1\alpha}(\partial_\rho\psi\sigma^\rho\bar{\varepsilon}_2)\right] \Leftarrow \partial_\rho\psi\sigma^\rho = 0 \text{ (Eq. of motion)} \\ &= -\partial_\rho\psi_\alpha(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2)\end{aligned}$$

以上から

$$\begin{aligned}[\delta_1, \delta_2]\psi_\alpha &= -2i(\sigma^\mu\bar{\varepsilon}_2)_\alpha(\varepsilon_1^\beta\partial_\mu\psi_\beta) - (1 \leftrightarrow 2) = -2i(\sigma^\mu\bar{\varepsilon}_2)_\alpha(\varepsilon_1\partial_\mu\psi) - (1 \leftrightarrow 2) \\ &\Leftarrow (\varepsilon_1\partial_\rho\psi)(\sigma^\rho\bar{\varepsilon}_2)_\alpha = -\partial_\rho\psi_\alpha(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2) \\ &= -2i\left[-\partial_\rho\psi_\alpha(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2)\right] - (1 \leftrightarrow 2) = 2i(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2)\partial_\rho\psi_\alpha - (1 \leftrightarrow 2) = 2i(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2 - \varepsilon_2\sigma^\rho\bar{\varepsilon}_1)\partial_\rho\psi_\alpha \\ &= 2i(\varepsilon_1\sigma^\rho\bar{\varepsilon}_2 + \bar{\varepsilon}_1\bar{\sigma}^\rho\varepsilon_2)\partial_\rho\psi_\alpha\end{aligned}$$

つまり、

$$\begin{aligned}[\delta_1, \delta_2]\phi &= 2i(\bar{\varepsilon}_1\bar{\sigma}^\mu\varepsilon_2 + \varepsilon_1\sigma^\mu\bar{\varepsilon}_2)\partial_\mu\phi \\ [\delta_1, \delta_2]\psi_\alpha &= 2i(\bar{\varepsilon}_1\bar{\sigma}^\mu\varepsilon_2 + \varepsilon_1\sigma^\mu\bar{\varepsilon}_2)\partial_\mu\psi_\alpha\end{aligned}$$

さて、超対称性の generator  $\delta A = [i\varepsilon S + i\bar{\varepsilon}\bar{S}, A]$  を使用すると

$$\begin{aligned}[\delta_1, \delta_2]A &= \delta_1\left([\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, A]\right) - (1 \leftrightarrow 2) = [\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, [\varepsilon_1 S + \bar{\varepsilon}_1\bar{S}, A]] - (1 \leftrightarrow 2) \\ &= [\varepsilon_1 S + \bar{\varepsilon}_1\bar{S}, [\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, A]] - [\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, [\varepsilon_1 S + \bar{\varepsilon}_1\bar{S}, A]] \\ &= [\varepsilon_1 S + \bar{\varepsilon}_1\bar{S}, [\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, A]] + [\varepsilon_2 S + \bar{\varepsilon}_2\bar{S}, [A, \varepsilon_1 S + \bar{\varepsilon}_1\bar{S}]]\end{aligned}$$

ここで Jacobi identity

$$\left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \left[ \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S}, A \right] \right] + \left[ \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S}, \left[ A, \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S} \right] \right] + \left[ A, \left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right] \right] = 0$$

を使用すると

$$[\delta_1, \delta_2] A = - \left[ A, \left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right] \right] = \left[ \left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right], A \right]$$

になる。一方、

$$[\delta_1, \delta_2] A = 2i \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) \partial_\mu A = 2 \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) \left[ P_\mu, A \right]$$

と比較すれば、

$$[\delta_1, \delta_2] A = \left[ \left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right], A \right] = 2 \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) \left[ P_\mu, A \right]$$

より、

$$\left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right] = 2 \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) P_\mu$$

$$\rightarrow \left[ \varepsilon_1 S, \bar{\varepsilon}_2 \bar{S} \right] = 2 \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 P_\mu = -2 \bar{\varepsilon}_2 \bar{\sigma}^\mu \varepsilon_1 P_\mu$$

$$\rightarrow \left[ \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S \right] = 2 \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 P_\mu = -2 \varepsilon_2 \sigma^\mu \bar{\varepsilon}_1 P_\mu$$

$$\rightarrow \left[ \left[ \varepsilon_1 S, \varepsilon_2 S \right], A \right] = \left[ \left[ \bar{\varepsilon}_1 \bar{S}, \bar{\varepsilon}_2 \bar{S} \right], A \right] = 0$$

$$\left[ \varepsilon_1 S, \bar{\varepsilon}_2 \bar{S} \right] = \left[ \varepsilon_1^\alpha S_\alpha, \bar{\varepsilon}_2^\beta \bar{S}_\beta \right]$$

$$= \varepsilon_1^\alpha \{ S_\alpha, \bar{S}_\beta \} \bar{\varepsilon}_2^\beta = 2 \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 P_\mu \Rightarrow \{ S_\alpha, \bar{S}_\beta \} = 2 \left( \sigma^\mu \right)_{\alpha\beta} P_\mu$$

$$\left[ \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S \right] = - \left[ \bar{\varepsilon}_2^\beta \bar{S}_\beta, S^\alpha \varepsilon_{1\alpha} \right]$$

$$= - \bar{\varepsilon}_2^\beta \{ \bar{S}_\beta, S^\alpha \} \varepsilon_{1\alpha} = -2 \bar{\varepsilon}_2 \bar{\sigma}^\mu \varepsilon_1 P_\mu \Rightarrow \{ S^\alpha, \bar{S}^\beta \} = 2 \left( \bar{\sigma}^\mu \right)^{\beta\alpha} P_\mu$$

$$\left[ \left[ \varepsilon_2 S, \varepsilon_1 S \right], A \right] = \left[ \left[ \varepsilon_2 \bar{S}, \varepsilon_1 \bar{S} \right], A \right] = 0 \Rightarrow \{ S^\alpha, S^\beta \} = \{ \bar{S}^\alpha, \bar{S}^\beta \} = 0$$

以上から、

$$\{ S_\alpha, \bar{S}_\beta \} = 2 \left( \sigma^\mu \right)_{\alpha\beta} P_\mu, \quad \{ S^\alpha, \bar{S}^\beta \} = 2 \left( \bar{\sigma}^\mu \right)^{\alpha\beta} P_\mu, \quad \{ S^\alpha, S^\beta \} = \{ \bar{S}^\alpha, \bar{S}^\beta \} = 0$$

$$\left[ \left[ \varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S} \right], A \right] = 2i \left( \bar{\varepsilon}_1 \bar{\sigma}^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \right) \partial_\mu A$$

$$\delta A = \left[ \varepsilon S + \bar{\varepsilon} \bar{S}, A \right]$$

$$\delta \phi = \left[ \varepsilon S, \phi \right] = \sqrt{2} \varepsilon \psi$$

$$\delta \psi_\alpha = \left[ \bar{\varepsilon} \bar{S}, \psi_\alpha \right] = -\sqrt{2} i \left( \sigma^\mu \bar{\varepsilon} \right)_\alpha \partial_\mu \phi, \quad \delta \psi^\alpha = \left[ \bar{\varepsilon} \bar{S}, \psi^\alpha \right] = \sqrt{2} i \left( \bar{\varepsilon} \bar{\sigma}^\mu \right)^\alpha \partial_\mu \phi$$

$$\delta \bar{\psi}_\alpha = \left[ \varepsilon S, \bar{\psi}_\alpha \right] = \sqrt{2} i \left( \varepsilon \sigma^\mu \right)_\alpha \partial_\mu \phi^\dagger, \quad \delta \bar{\psi}^\alpha = \left[ \varepsilon S, \bar{\psi}^\alpha \right] = -\sqrt{2} i \left( \bar{\sigma}^\mu \varepsilon \right)^\alpha \partial_\mu \phi^\dagger$$

## 補助場の変換性

残っている  $[\varepsilon S, \psi_\alpha]$  が新たな補助場  $F$  に変換されるとして調べる。

$$\begin{aligned}\delta\psi &= [\varepsilon S + \bar{\varepsilon}\bar{S}, \psi] \\ \Rightarrow \delta\psi_\alpha &= [\bar{\varepsilon}\bar{S}, \psi_\alpha] = -\sqrt{2}i(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi, \quad \delta\psi^\alpha = [\bar{\varepsilon}\bar{S}, \psi^\alpha] = \sqrt{2}i(\bar{\varepsilon}\sigma^\mu)^\alpha \partial_\mu \phi \\ \Rightarrow \delta\psi_\alpha &= [\varepsilon S, \psi_\alpha] = ??? \rightarrow \text{assume } \{S_\alpha, \psi_\beta\} = -\sqrt{2}F_{\alpha\beta} \Leftrightarrow [\varepsilon S, \psi_\beta] = -\sqrt{2}\varepsilon^\alpha F_{\alpha\beta}\end{aligned}$$

ここで先ほどの Jacobi 恒等式から導出された

$$[[\varepsilon_1 S + \bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S + \bar{\varepsilon}_2 \bar{S}], A] = 2i(\bar{\varepsilon}_1 \sigma^\mu \varepsilon_2 + \varepsilon_1 \sigma^\mu \bar{\varepsilon}_2) \partial_\mu A$$

を用いて、

$$\begin{aligned}[[\varepsilon_1 S, \bar{\varepsilon}_2 \bar{S}], A] &= 2i\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu A, \quad [[\bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S], A] = 2i\bar{\varepsilon}_1 \sigma^\mu \varepsilon_2 \partial_\mu A \\ [[\varepsilon_1 S, \bar{\varepsilon}_2 \bar{S}], A] &= [\varepsilon_1 S, [\bar{\varepsilon}_2 \bar{S}, A]] + [\bar{\varepsilon}_2 \bar{S}, [A, \varepsilon_1 S]] = 2i\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu A \\ [[\bar{\varepsilon}_1 \bar{S}, \varepsilon_2 S], A] &= [\bar{\varepsilon}_1 \bar{S}, [\varepsilon_2 S, A]] + [\varepsilon_2 S, [A, \bar{\varepsilon}_1 \bar{S}]] = 2i\bar{\varepsilon}_1 \sigma^\mu \varepsilon_2 \partial_\mu A\end{aligned}$$

に、 $A = \psi$  として

$$\begin{aligned}[[\varepsilon_1 S, \bar{\varepsilon}_2 \bar{S}], \psi_\alpha] &= [\varepsilon_1 S, [\bar{\varepsilon}_2 \bar{S}, \psi_\alpha]] + [\bar{\varepsilon}_2 \bar{S}, [\psi_\alpha, \varepsilon_1 S]] \\ &= [\varepsilon_1 S, [\bar{\varepsilon}_2 \bar{S}, \psi_\alpha]] + [\bar{\varepsilon}_2 \bar{S}, -[\varepsilon_1 S, \psi_\alpha]] = 2i\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu \psi_\alpha \\ \Leftrightarrow [\varepsilon_1 S, \psi_\beta] &= -\sqrt{2}\varepsilon_1^\alpha F_{\alpha\beta} \rightarrow [\varepsilon_1 S, \psi_\alpha] = -\sqrt{2}\varepsilon_1^\beta F_{\beta\alpha} \\ [\varepsilon_1 S, (-\sqrt{2}i(\sigma^\mu \bar{\varepsilon}_2)_\alpha \partial_\mu \phi)] &+ [\bar{\varepsilon}_2 \bar{S}, -(-\sqrt{2}\varepsilon_1^\beta F_{\beta\alpha})] = 2i\varepsilon_1 \sigma^\mu \bar{\varepsilon}_2 \partial_\mu \psi_\alpha \\ -\sqrt{2}i(\sigma^\mu \bar{\varepsilon}_2)_\alpha \partial_\mu (\sqrt{2}\varepsilon_1^\beta \psi) &+ \varepsilon_1^\beta [\bar{\varepsilon}_2 \bar{S}, \sqrt{2}F_{\beta\alpha}] = 2i\varepsilon_1^\beta \sigma^\mu \bar{\varepsilon}_2 \partial_\mu \psi_\alpha \\ -\sqrt{2}i(\sigma^\mu)_{\alpha\beta'} \bar{\varepsilon}_2^{\beta'} \partial_\mu (\sqrt{2}\varepsilon_1^\beta \psi_\beta) &+ \varepsilon_1^\beta [-\bar{\varepsilon}_2^{\beta'} \bar{S}_{\beta'}, \sqrt{2}F_{\beta\alpha}] = 2i\varepsilon_1^\beta (\sigma^\mu)_{\beta\beta'} \bar{\varepsilon}_2^{\beta'} \partial_\mu \psi_\alpha \\ -\sqrt{2}i(\sigma^\mu)_{\alpha\beta'} \partial_\mu (\sqrt{2}\varepsilon_1^\beta \psi_\beta) \bar{\varepsilon}_2^{\beta'} &+ \varepsilon_1^\beta [\bar{S}_{\beta'}, \sqrt{2}F_{\beta\alpha}] \bar{\varepsilon}_2^{\beta'} = -2i\varepsilon_1^\beta (\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \bar{\varepsilon}_2^{\beta'} \\ -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta &+ [\bar{S}_{\beta'}, \sqrt{2}F_{\beta\alpha}] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \\ \Rightarrow [\bar{S}_{\beta'}, F_{\beta\alpha}] &= \sqrt{2}i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta - \sqrt{2}i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \\ \xrightarrow{\alpha \leftrightarrow \beta} [\bar{S}_{\beta'}, F_{\alpha\beta}] &= \sqrt{2}i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha - \sqrt{2}i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta \\ \Rightarrow [\bar{S}_{\beta'}, F_{\alpha\beta}] &+ [\bar{S}_{\beta'}, F_{\beta\alpha}] = 0 \Rightarrow F_{\alpha\beta} + F_{\beta\alpha} = 0 \Rightarrow F_{\alpha\beta} = \varepsilon_{\alpha\beta} F \\ \Rightarrow [\varepsilon_1 S, \psi_\beta] &= -\sqrt{2}\varepsilon_1^\alpha F_{\alpha\beta} = -\sqrt{2}\varepsilon_1^\alpha \varepsilon_{\alpha\beta} F = \sqrt{2}\varepsilon_1^\alpha \varepsilon_{\beta\alpha} F = \sqrt{2}\varepsilon_{1\beta} F \quad (\varepsilon_{1\beta} \equiv \varepsilon_{\beta\alpha} \varepsilon_1^\alpha)\end{aligned}$$

ゆえに、

$$[\varepsilon_1 S, \psi_\beta] = -\sqrt{2}\varepsilon_1^\alpha F_{\alpha\beta} = \sqrt{2}\varepsilon_{1\beta} F$$

次に、この新たな場  $F$  に対する超変換性を調べる。 $A = \psi$  の結果を使用すると

$$\begin{aligned}
& -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta + [\bar{S}_{\beta'}, \sqrt{2}F_{\beta\alpha}] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \\
& -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta + [\bar{S}_{\beta'}, \sqrt{2}\varepsilon_{\beta\alpha}F] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \\
& -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi_\beta + \sqrt{2}\varepsilon_{\beta\alpha} [\bar{S}_{\beta'}, F] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \psi_\alpha \\
& \quad \Leftarrow \varepsilon^{\alpha\beta} \varepsilon_{\beta\alpha'} = \delta_{\alpha'}^\alpha (\varepsilon^{12} = \varepsilon_{21} = 1), \quad \psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta \\
& -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \varepsilon^{\alpha\beta} \psi_\beta + \sqrt{2}\varepsilon^{\alpha\beta} \varepsilon_{\beta\alpha} [\bar{S}_{\beta'}, F] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \varepsilon^{\alpha\beta} \psi_\alpha = 2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \varepsilon^{\beta\alpha} \psi_\alpha \\
& -2i(\sigma^\mu)_{\alpha\beta'} \partial_\mu \psi^\alpha + \sqrt{2}2[\bar{S}_{\beta'}, F] = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu \varepsilon^{\alpha\beta} \psi_\alpha = -2i(\sigma^\mu)_{\beta\beta'} \partial_\mu (-\psi^\beta) \\
& -2i(\partial_\mu \psi \sigma^\mu)_{\beta'} + 2\sqrt{2}[\bar{S}_{\beta'}, F] = 2i(\partial_\mu \psi \sigma^\mu)_{\beta'} \Rightarrow 2\sqrt{2}[\bar{S}_{\beta'}, F] = 4i(\partial_\mu \psi \sigma^\mu)_{\beta'} \\
& \quad \Rightarrow \boxed{[\bar{S}_{\beta'}, F] = \sqrt{2}(\partial_\mu \psi \sigma^\mu)_{\beta'}}
\end{aligned}$$

同様に、Jacobi identity :

$$[\varepsilon_1 S, [\varepsilon_2 S, A]] + [\varepsilon_2 S, [A, \varepsilon_1 S]] = -[A, [\varepsilon_1 S, \varepsilon_2 S]] = 0$$

より

$$\begin{aligned}
& [\varepsilon_1 S, [\varepsilon_2 S, \psi_\beta]] - [\varepsilon_2 S, [\varepsilon_1 S, \psi_\beta]] = 0 \Leftarrow [\varepsilon_2 S, \psi_\beta] = -\sqrt{2}\varepsilon_2^\alpha F_{\alpha\beta} = \sqrt{2}\varepsilon_{2\beta} F \\
& [\varepsilon_1^{\alpha'} S_{\alpha'}, \sqrt{2}\varepsilon_{2\beta} F] - [\varepsilon_2^{\alpha'} S_{\alpha'}, \sqrt{2}\varepsilon_{1\beta} F] = 0 \\
& \varepsilon_1^{\alpha'} \varepsilon_{2\beta} [S_{\alpha'}, F] - \varepsilon_2^{\alpha'} \varepsilon_{1\beta} [S_{\alpha'}, F] = (\varepsilon_1^{\alpha'} \varepsilon_{2\beta} - \varepsilon_2^{\alpha'} \varepsilon_{1\beta}) [S_{\alpha'}, F] = 0 \\
& \quad \Rightarrow \boxed{[S_{\alpha'}, F] = 0}
\end{aligned}$$

以上から

$$\begin{aligned}
& [S_\alpha, F] = 0, \quad [\bar{S}_\alpha, F] = \sqrt{2}(\partial_\mu \psi \sigma^\mu)_\alpha \Rightarrow [\varepsilon S, F] = 0, \quad [\bar{S}\bar{\varepsilon}, F] = [\bar{\varepsilon}\bar{S}, F] = \sqrt{2}i\partial_\mu \psi \sigma^\mu \bar{\varepsilon} \\
& \Rightarrow \delta F = [\varepsilon S + \bar{\varepsilon}\bar{S}, F] = \sqrt{2}i\partial_\mu \psi \sigma^\mu \bar{\varepsilon}
\end{aligned}$$

および

$$\begin{aligned}
& [\varepsilon S, \psi_\alpha] = \sqrt{2}\varepsilon_\alpha F \oplus \delta\psi_\alpha = -\sqrt{2}i(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi \\
& \Rightarrow \delta\psi_\alpha = [\varepsilon S + \bar{\varepsilon}\bar{S}, \psi_\alpha] = -\sqrt{2}i(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi + \sqrt{2}\varepsilon_\alpha F
\end{aligned}$$

以上をまとめると、

$$\begin{aligned}
& \delta\phi = \sqrt{2}\varepsilon\psi, \quad \delta\phi^\dagger = \sqrt{2}\bar{\varepsilon}\bar{\psi} \\
& \delta\psi_\alpha = -\sqrt{2}i(\sigma^\mu \bar{\varepsilon})_\alpha \partial_\mu \phi + \sqrt{2}\varepsilon_\alpha F, \quad \delta\psi^\alpha = \sqrt{2}i(\bar{\varepsilon}\sigma^\mu)^\alpha \partial_\mu \phi + \sqrt{2}\varepsilon^\alpha F \\
& \delta\bar{\psi}_\alpha = \sqrt{2}i(\varepsilon\sigma^\mu)_\alpha \partial_\mu \phi^\dagger + \sqrt{2}\bar{\varepsilon}_\alpha F^\dagger, \quad \delta\bar{\psi}^\alpha = -\sqrt{2}i(\bar{\sigma}^\mu \varepsilon)^\alpha \partial_\mu \phi^\dagger + \sqrt{2}\bar{\varepsilon}^\alpha F^\dagger \\
& \delta F = \sqrt{2}i\partial_\mu \psi \sigma^\mu \bar{\varepsilon}, \quad \delta F^\dagger = -\sqrt{2}i\varepsilon\bar{\sigma}^\mu \partial_\mu \bar{\psi} = \sqrt{2}\partial_\mu \bar{\psi} i\sigma^\mu \varepsilon \Leftarrow (\alpha\sigma^\mu \bar{\beta})^\dagger = \beta\bar{\sigma}^\mu \bar{\alpha} = -\bar{\alpha}\sigma^\mu \beta
\end{aligned}$$

になる。