

第7回: 角運動量保存則

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必要になる基礎知識

○半径 r [m]、中心角 θ [ラジアン]の扇形の円弧の長さは？

○ベクトルの外積、ベクトル $a \times b$ の方向を説明すると？

○ a と **b** の間の角度を θ とするとき $a \times b$ の長さ(大きさ)は $|a||b|$ を含む式で？

$$a \times \frac{db}{dt} + \frac{da}{dt} \times b = \frac{d}{dt} \boxed{\quad}$$

○運動量ベクトル p を速度を含む式で表すと？

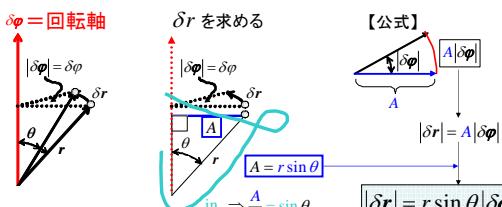
○角運動量ベクトル L を位置ベクトル r と運動量ベクトル p で表すと？

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空間の等方性

$$\mathcal{L}(r'(t), v'(t), t) = \mathcal{L}(r(t), v(t), t) \leftarrow \begin{cases} r'(t) = \text{'r(t)を回転させる'} \\ v'(t) = \text{'v(t)を同時に同じだけ回転させる'} \end{cases}$$

回転をベクトルの外積で表すと…



$$\boxed{|\delta r| = r \sin \theta |\delta \phi|}$$

ベクトルの外積にsin関数あり

$\delta\phi = \text{回転軸}$

$$\begin{aligned} & a \times b = \delta\phi \times r \\ & a \times b = |a||b| \sin \theta \\ & \delta r = \delta\phi \times r \\ & \delta r = r \sin \theta |\delta\phi| \end{aligned}$$

r だけでなく、あらゆるベクトルに対して成立
ベクトルを代表して A とする

$\delta A = \delta\phi \times A \quad (A = r \nparallel v \text{ 等})$

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空間の等方性と保存則

$$\delta A = \delta\phi \times A \quad (A = r \nparallel v \text{ 等})$$

$$\mathcal{L}(r(t) + \delta\phi \times r(t), v(t) + \delta\phi \times v(t), t) = \mathcal{L}(r(t), v(t), t) \leftarrow \begin{cases} r'(t) = \text{'r(t)を回転させる'} \\ v'(t) = \text{'v(t)を同時に同じだけ回転させる'} \end{cases}$$

$\delta r(t) = \delta\phi \times r(t) \xrightarrow{\delta r = r'(t) - r(t)} r'(t) = r(t) + \delta\phi \times r(t)$

$\delta v(t) = \delta\phi \times v(t) \xrightarrow{\delta v = v'(t) - v(t)} v'(t) = v(t) + \delta\phi \times v(t)$

$$\mathcal{L}(r(t) + \delta\phi \times r(t), v(t) + \delta\phi \times v(t), t) = \mathcal{L}(r(t), v(t), t)$$

$$\mathcal{L}(r(t) + \delta\phi \times r(t), v(t) + \delta\phi \times v(t), t) = \mathcal{L}(r(t), v(t), t)$$

ベクトル

ベクトルの内積

$$\begin{aligned} \text{【公式】 } f(r(t) + a) &\stackrel{a = (a_x, a_y, a_z)}{=} f(r(t)) + \underbrace{\overbrace{a}^{\text{ベクトル}} \cdot \overbrace{\frac{\partial f(r(t))}{\partial r(t)}}^{\text{ベクトル}}}_{\text{ベクトルの内積}} \\ &= f(r(t)) + a_x \frac{\partial f(r(t))}{\partial x(t)} + a_y \frac{\partial f(r(t))}{\partial y(t)} + a_z \frac{\partial f(r(t))}{\partial z(t)} \\ f(v(t) + b) &\stackrel{b = (b_x, b_y, b_z)}{=} f(v(t)) + b \cdot \frac{\partial f(v(t))}{\partial v(t)} \\ &= f(v(t)) + b_x \frac{\partial f(v(t))}{\partial v_x(t)} + b_y \frac{\partial f(v(t))}{\partial v_y(t)} + b_z \frac{\partial f(v(t))}{\partial v_z(t)} \\ f(r(t) + a, v(t) + b) &\stackrel{a = (a_x, a_y, a_z), b = (b_x, b_y, b_z)}{=} f(r(t), v(t)) + a \cdot \frac{\partial f(r(t), v(t))}{\partial r(t)} + b \cdot \frac{\partial f(r(t), v(t))}{\partial v(t)} \\ &= f(r(t), v(t)) + a_x \frac{\partial f(r(t), v(t))}{\partial x(t)} + a_y \frac{\partial f(r(t), v(t))}{\partial y(t)} + a_z \frac{\partial f(r(t), v(t))}{\partial z(t)} \\ &\quad + b_x \frac{\partial f(r(t), v(t))}{\partial v_x(t)} + b_y \frac{\partial f(r(t), v(t))}{\partial v_y(t)} + b_z \frac{\partial f(r(t), v(t))}{\partial v_z(t)} \end{aligned}$$

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$$\begin{aligned} \mathcal{L}(r(t) + \delta\varphi \times r(t), v(t) + \delta\varphi \times v(t), t) &= \mathcal{L}(r(t), v(t), t) \\ \mathcal{L}(r(t) + a, v(t) + b) &\stackrel{\text{def}}{=} \mathcal{L}(r(t), v(t)) + a \cdot \frac{\partial \mathcal{L}(r(t), v(t))}{\partial r(t)} + b \cdot \frac{\partial \mathcal{L}(r(t), v(t))}{\partial v(t)} \end{aligned}$$

$$\begin{aligned} &\left(r(t) + \underbrace{\delta\varphi \times r(t)}_a, v(t) + \underbrace{\delta\varphi \times v(t)}_b, t \right) = 0 \\ &\stackrel{\text{def}}{=} \mathcal{L}(r(t), v(t)) + \underbrace{\delta\varphi \times r(t)}_a \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + \underbrace{\delta\varphi \times v(t)}_b \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \\ &= \mathcal{L}(r(t), v(t), t) \end{aligned}$$

$$(\delta\varphi \times r(t)) \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + (\delta\varphi \times v(t)) \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} = 0$$

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$$(r(t) \times \delta\varphi) \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + (v(t) \times \delta\varphi) \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} = 0$$

3つのベクトルのかかけ算…

$$\begin{aligned} &\left(\underbrace{\frac{a}{\delta\varphi} \times r(t)}_{\delta\varphi \times r(t)}, \underbrace{\frac{b}{\delta\varphi} \times v(t)}_{\delta\varphi \times v(t)} \right) \cdot \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial \delta\varphi} = 0 \\ &\Leftrightarrow (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \\ &\left(\frac{b}{r(t)} \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} \right) \cdot \frac{a}{\delta\varphi} + \left(\frac{a}{v(t)} \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \right) \cdot \frac{b}{\delta\varphi} = 0 \\ &\left(\frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + v(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \right) \cdot \delta\varphi = 0 \\ &r(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + v(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} = \overset{\text{零ベクトル}}{0} \end{aligned}$$

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$$r(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + v(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} = \overset{\text{零ベクトル}}{0}$$

ラグランジュの運動方程式…

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \right) &= \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} \\ \overset{\text{零ベクトル}}{0} &= r(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial r(t)} + v(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \\ &= r(t) \times \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) + v(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)} \\ &= \underbrace{r(t) \times \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right)}_a + \underbrace{\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right)}_b \\ &\quad \text{【公式】 } a \times \frac{db}{dt} + \frac{da}{dt} \times b = \frac{d}{dt} (a \times b) \\ &= \underbrace{\frac{d}{dt} \left(r(t) \times \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right)}_L \overset{\text{Lと書く}}{=} \underbrace{\frac{d}{dt} \left(r(t) \times \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right)}_{\overset{\text{零ベクトル}}{0}} \end{aligned}$$

$$L = r(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)}$$

は保存する $\Leftrightarrow \frac{dL}{dt} = \overset{\text{零ベクトル}}{0}$

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角運動量(L と書く)保存則

$$L = r(t) \times \frac{\partial \mathcal{L}(r(t), v(t), t)}{\partial v(t)}$$

ニュートン力学の場合に、角運動量に一致するはず！！！

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(x(t), v(t)) &= \frac{1}{2} m [\mathbf{v}(t)]^2 - U(x(t)) \\ L &= r(t) \times \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} \\ &\Leftarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} = \frac{\partial}{\partial v(t)} \left[\frac{1}{2} m [\mathbf{v}(t)]^2 \right] = \overset{\text{運動量}}{\frac{\partial}{\partial v(t)}} \mathbf{v}(t) = p(t) \\ &= r(t) \times p(t) \\ L &= r(t) \times \frac{\partial \mathcal{L}(r(t), v(t))}{\partial v(t)} \text{ は角運動量であり保存する} \end{aligned}$$

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N粒子系の角運動量

$$L = \sum_{i=1}^N r_i(t) \times \frac{\partial \mathcal{L}(r_1(t), r_2(t), \dots, r_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)}$$

ニュートン力学の場合に、これは全角運動量？

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(r_1(t), r_2(t), \dots, r_N(t), v_1(t), v_2(t), \dots, v_N(t)) &= \frac{1}{2} \sum_{i=1}^N m_i [\mathbf{v}_i(t)]^2 - U(r_1(t), r_2(t), \dots, r_N(t)) \\ L &= \sum_{i=1}^N r_i \times \frac{\partial \mathcal{L}_{\text{ニュートン}}(r_1(t), r_2(t), \dots, r_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} \\ &\stackrel{\text{全角運動量に一致}}{=} \frac{\partial \mathcal{L}_{\text{ニュートン}}(r_1(t), r_2(t), \dots, r_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} = \frac{\partial}{\partial v_i(t)} \left[\frac{1}{2} \sum_{i=1}^N m_i [\mathbf{v}_i(t)]^2 \right] = \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial [\mathbf{v}_i(t)]^2}{\partial v_i(t)} \\ &= \frac{1}{2} \sum_{i=1}^N m_i 2\mathbf{v}_i(t) \delta_{ii} = m_i \mathbf{v}_i(t) = p_i(t) \\ &= \sum_{i=1}^N r_i(t) \times p_i(t) \\ L &= \sum_{i=1}^N r_i(t) \times \frac{\partial \mathcal{L}(r_1(t), r_2(t), \dots, r_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} \end{aligned}$$

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