

第7回：角運動量保存則

必要になる基礎知識

- 半径 r [m]、中心角 θ [ラジアン]の扇形の円弧の長さは？
- ベクトルの外積、ベクトル $\mathbf{a} \times \mathbf{b}$ の方向を説明すると？
- \mathbf{a} と \mathbf{b} の間の角度を θ とするとき $\mathbf{a} \times \mathbf{b}$ の長さ(大きさ)は $|\mathbf{a}|, |\mathbf{b}|$ を含む式で？

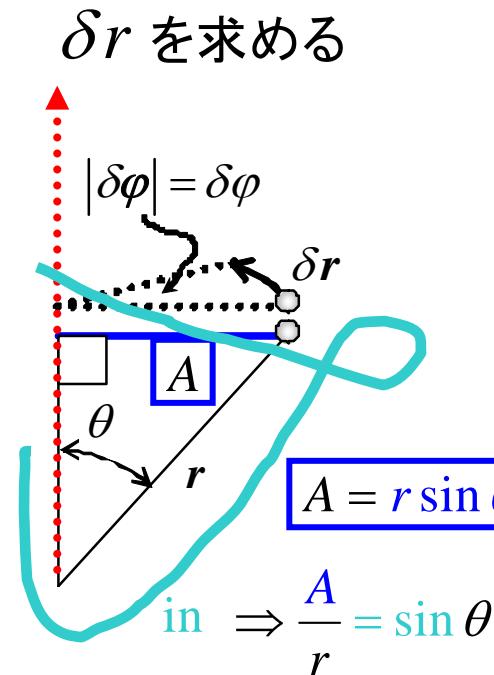
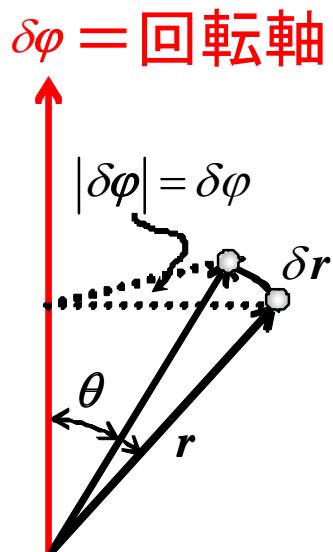
$$\textcircled{○} \mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b} = \frac{d}{dt} \boxed{\quad}$$

- 運動量ベクトル \mathbf{p} を速度を含む式で表すと？
- 角運動量 \mathbf{L} を位置ベクトル \mathbf{r} と運動量ベクトル \mathbf{p} で表すと？

空間の等方性

$\mathcal{L}(r'(t), v'(t), t) = \mathcal{L}(r(t), v(t), t) \Leftarrow \begin{cases} r'(t) = 「r(t)を回転させる」 \\ v'(t) = 「v(t)を同時に同じだけ回転させる」 \end{cases}$

回転をベクトルの外積で表すと…



【公式】

A

$A |\delta\varphi|$

$|delta r| = A |\delta\varphi|$

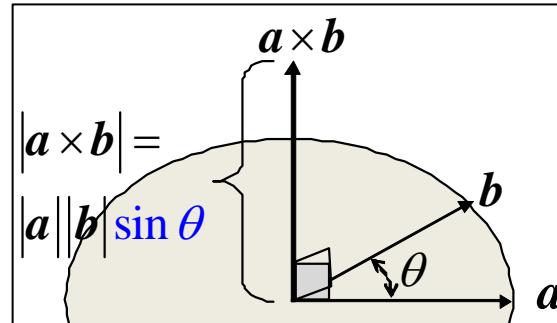
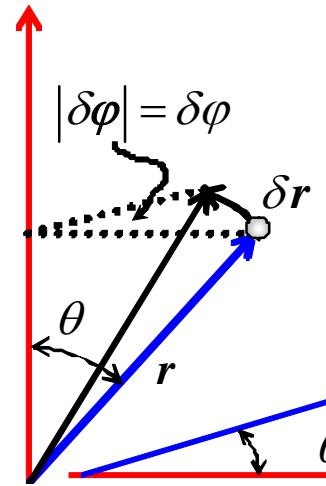
$A = r \sin \theta$

$|delta r| = r \sin \theta |\delta\varphi|$

$$|\delta r| = r \sin \theta |\delta\varphi|$$

ベクトルの外積にsin関数あり

$\delta\varphi$ = 回転軸



$$a \times b = \delta\varphi \times r$$

$$a = \delta\varphi$$

$$b = r$$

=

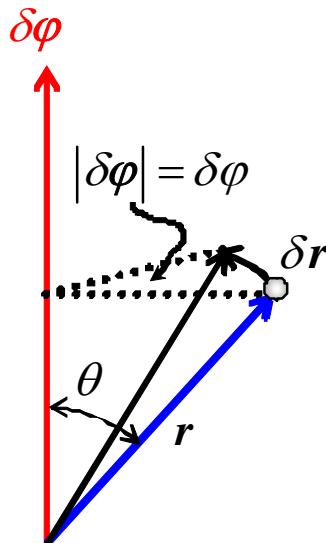
$$\delta r = \overbrace{\delta\varphi \times r}^{a \times b}$$

$$\delta r = \delta\varphi \times r$$

r だけでなく、あらゆるベクトルに対して成立
ベクトルを代表して A とする

$$\delta A = \delta\varphi \times A \quad (A = r \text{ や } v \text{ 等})$$

空間の等方性と保存則



$$\delta A = \delta\phi \times A \quad (A = \mathbf{r} \text{ や } \mathbf{v} \text{ 等})$$

$\mathcal{L}(\mathbf{r}'(t), \mathbf{v}'(t), t) = \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t) \Leftarrow \begin{cases} \mathbf{r}'(t) = \text{「}\mathbf{r}(t)\text{を回転させる」} \\ \mathbf{v}'(t) = \text{「}\mathbf{v}(t)\text{を同時に同じ} \\ \text{だけ回転させる」} \end{cases}$



$$\delta \mathbf{r}(t) = \delta\phi \times \mathbf{r}(t) \stackrel{\delta \mathbf{r} = \mathbf{r}'(t) - \mathbf{r}(t)}{\Rightarrow} \mathbf{r}'(t) = \mathbf{r}(t) + \delta\phi \times \mathbf{r}(t)$$

$$\delta \mathbf{v}(t) = \delta\phi \times \mathbf{v}(t) \stackrel{\delta \mathbf{v} = \mathbf{v}'(t) - \mathbf{v}(t)}{\Rightarrow} \mathbf{v}'(t) = \mathbf{v}(t) + \delta\phi \times \mathbf{v}(t)$$



$$\mathcal{L}(\mathbf{r}(t) + \delta\phi \times \mathbf{r}(t), \mathbf{v}(t) + \delta\phi \times \mathbf{v}(t), t) = \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)$$

$$\mathcal{L}(\mathbf{r}(t) + \delta\varphi \times \mathbf{r}(t), \mathbf{v}(t) + \delta\varphi \times \mathbf{v}(t), \mathbf{t}) = \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})$$

【公式】 $f(\mathbf{r}(t) + \mathbf{a}) \doteq f(\mathbf{r}(t)) + \underbrace{\hat{\mathbf{a}} \cdot \overbrace{\frac{\partial f(\mathbf{r}(t))}{\partial \mathbf{r}(t)}}^{\text{ベクトル}}}_{\text{ベクトルの内積}}$

$\mathbf{a} = (a_x, a_y, a_z)$
 $\mathbf{r}(t) = (x(t), y(t), z(t))$

$$= f(\mathbf{r}(t)) + a_x \frac{\partial f(\mathbf{r}(t))}{\partial \mathbf{x}(t)} + a_y \frac{\partial f(\mathbf{r}(t))}{\partial \mathbf{y}(t)} + a_z \frac{\partial f(\mathbf{r}(t))}{\partial \mathbf{z}(t)}$$

$f(\mathbf{v}(t) + \mathbf{b}) \doteq f(\mathbf{v}(t)) + \mathbf{b} \cdot \frac{\partial f(\mathbf{v}(t))}{\partial \mathbf{v}(t)}$

$\mathbf{b} = (b_x, b_y, b_z)$
 $\mathbf{v}(t) = (v_x(t), v_y(t), v_z(t))$

$$= f(\mathbf{v}(t)) + b_x \frac{\partial f(\mathbf{v}(t))}{\partial \mathbf{v}_x(t)} + b_y \frac{\partial f(\mathbf{v}(t))}{\partial \mathbf{v}_y(t)} + b_z \frac{\partial f(\mathbf{v}(t))}{\partial \mathbf{v}_z(t)}$$

$f(\mathbf{r}(t) + \mathbf{a}, \mathbf{v}(t) + \mathbf{b}) \doteq f(\mathbf{r}(t), \mathbf{v}(t)) + \mathbf{a} \cdot \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{r}(t)} + \mathbf{b} \cdot \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}(t)}$

$$= f(\mathbf{r}(t), \mathbf{v}(t)) + a_x \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{x}(t)} + a_y \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{y}(t)} + a_z \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{z}(t)}$$

$$+ b_x \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}_x(t)} + b_y \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}_y(t)} + b_z \frac{\partial f(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}_z(t)}$$

$$\mathcal{L}(\mathbf{r}(t) + \delta\varphi \times \mathbf{r}(t), \mathbf{v}(t) + \delta\varphi \times \mathbf{v}(t), \textcolor{red}{t}) = \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t})$$

$$\mathcal{L}(\mathbf{r}(t) + \mathbf{a}, \mathbf{v}(t) + \mathbf{b}) \doteq \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t)) + \mathbf{a} \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{r}(t)} + \mathbf{b} \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}(t)}$$

$$\begin{aligned} & \mathcal{L}\left(\mathbf{r}(t) + \overbrace{\delta\varphi \times \mathbf{r}(t)}^a, \mathbf{v}(t) + \overbrace{\delta\varphi \times \mathbf{v}(t)}^b, \textcolor{red}{t}\right) \\ & \qquad \qquad \qquad = 0 \\ & \doteq \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t}) + \overbrace{\delta\varphi \times \mathbf{r}(t)}^a \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t})}{\partial \mathbf{r}(t)} + \overbrace{\delta\varphi \times \mathbf{v}(t)}^b \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t})}{\partial \mathbf{v}(t)} \\ & = \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t}) \end{aligned}$$

$$(\delta\varphi \times \mathbf{r}(t)) \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t})}{\partial \mathbf{r}(t)} + (\delta\varphi \times \mathbf{v}(t)) \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \textcolor{red}{t})}{\partial \mathbf{v}(t)} = 0$$

$$(\mathbf{r}(t) \times \delta\boldsymbol{\varphi}) \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{r}(t)} + (\mathbf{v}(t) \times \delta\boldsymbol{\varphi}) \cdot \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{v}(t)} = 0$$

3つのベクトルのかけ算…

$$\left(\frac{\overset{a}{\cancel{\delta\boldsymbol{\varphi}}} \times \overset{b}{\cancel{\mathbf{r}(t)}}}{\overset{c}{\cancel{\delta\boldsymbol{\varphi}}}} \right) \cdot \overbrace{\frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{r}(t)}}^c + \left(\frac{\overset{a}{\cancel{\delta\boldsymbol{\varphi}}} \times \overset{b}{\cancel{\mathbf{v}(t)}}}{\overset{c}{\cancel{\delta\boldsymbol{\varphi}}}} \right) \cdot \overbrace{\frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)}}^c = 0$$

$$\Leftrightarrow (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

$$\left(\overset{b}{\cancel{\mathbf{r}(t)}} \times \overbrace{\frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{r}(t)}}^c \right) \cdot \overset{a}{\cancel{\delta\boldsymbol{\varphi}}} + \left(\overset{b}{\cancel{\mathbf{v}(t)}} \times \overbrace{\frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)}}^c \right) \cdot \overset{a}{\cancel{\delta\boldsymbol{\varphi}}} = 0$$

$$\left(\overset{\text{ベクトル}}{\overbrace{\mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{r}(t)} + \mathbf{v}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{v}(t)}}} \right) \cdot \delta\boldsymbol{\varphi} = 0$$

$$\mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{r}(t)} + \mathbf{v}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), \mathbf{t})}{\partial \mathbf{v}(t)} = \overset{\text{零ベクトル}}{\hat{\theta}}$$

$$\mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{r}(t)} + \mathbf{v}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} = \overset{\text{零ベクトル}}{\hat{\theta}}$$

ラグランジュの運動方程式

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \right) = \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{r}(t)}$$

$$\begin{aligned} \overset{\text{零ベクトル}}{\hat{\theta}} &= \mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{r}(t)} + \mathbf{v}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \\ &= \mathbf{r}(t) \times \frac{d}{dt} \left(\frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \right) + \boxed{\mathbf{v}(t)} \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \\ &= \overbrace{\mathbf{r}(t) \times \frac{d}{dt} \left(\frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \right)}^{\mathbf{a}} + \boxed{\frac{d\mathbf{r}(t)}{dt}} \times \overbrace{\frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)}}^{\mathbf{b}} = \frac{d}{dt} (\mathbf{a} \times \mathbf{b}) \end{aligned}$$

【公式】 $\mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b} = \frac{d}{dt} (\mathbf{a} \times \mathbf{b})$

L と書く

$$= \frac{d}{dt} \left(\mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \right) \Rightarrow \overbrace{\frac{d}{dt} \left(\mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \right)}^{\text{零ベクトル}} = \overset{\text{零ベクトル}}{\hat{\theta}}$$

$$\mathbf{L} = \mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)} \text{ は保存する} \Leftrightarrow \frac{d\mathbf{L}}{dt} = \overset{\text{零ベクトル}}{\hat{\theta}}$$

角運動量(L と書く)保存則

$$L = \mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t), t)}{\partial \mathbf{v}(t)}$$

ニュートン力学の場合に、角運動量に一致するはず！！！

$$\begin{aligned}\mathcal{L}_{\text{ニュートン}}(\mathbf{x}(t), \mathbf{v}(t)) &= \frac{1}{2}m[\mathbf{v}(t)]^2 - U(\mathbf{x}(t)) \\ L &= \mathbf{r}(t) \times \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}(t), \mathbf{v}(t))}{\partial \mathbf{v}(t)} \\ &\Leftarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}(t), \mathbf{v}(t))}{\partial \mathbf{v}(t)} = \frac{\partial \frac{1}{2}m[\mathbf{v}(t)]^2}{\partial \mathbf{v}(t)} = \overbrace{m\mathbf{v}(t)}^{\text{運動量}} = \mathbf{p}(t) \\ &= \overbrace{\mathbf{r}(t) \times \mathbf{p}(t)}^{\text{角運動量}}\end{aligned}$$

$$L = \mathbf{r}(t) \times \frac{\partial \mathcal{L}(\mathbf{r}(t), \mathbf{v}(t))}{\partial \mathbf{v}(t)}$$
 は角運動量であり保存する

N粒子系の角運動量

$$\mathbf{L} = \sum_{i=1}^N \mathbf{r}_i(t) \times \frac{\partial \mathcal{L}(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)}$$

ニュートン力学の場合に、これは全角運動量？

$$\begin{aligned}
 \mathcal{L}_{\text{ニュートン}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t)) &= \frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2 - U(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t)) \\
 \mathbf{L} &= \sum_{i=1}^N \mathbf{r}_i \times \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} \\
 \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} &= \frac{\partial \frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2}{\partial \mathbf{v}_i(t)} = \frac{1}{2} \sum_{k=1}^N m_k \frac{\partial [\mathbf{v}_k(t)]^2}{\partial \mathbf{v}_i(t)} \\
 &= \frac{1}{2} \sum_{k=1}^N m_k 2\mathbf{v}_k(t) \delta_{ik} = m_i \mathbf{v}_i(t) = \mathbf{p}_i(t) \\
 &\stackrel{\text{全角運動量に一致}}{=} \sum_{i=1}^N \mathbf{r}_i(t) \times \mathbf{p}_i(t)
 \end{aligned}$$

$$\mathbf{L} = \sum_{i=1}^N \mathbf{r}_i(t) \times \frac{\partial \mathcal{L}(\mathbf{r}_1(t), \mathbf{r}_2(t), \dots, \mathbf{r}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)}$$