

第6回:エネルギー保存則・運動量保存則

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必要になる基礎知識

○ $f(x(t), t)$ のとき、 $\frac{df(x(t), t)}{dt}$ を $\frac{d}{dt} \frac{\partial}{\partial x}$ を含む式で表すと？

○ $\frac{dx^2}{dx}$ は？

○ $x = (x, y, z) = xi + yj + zk$ のとき $\frac{d}{dx}$ を $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ と*i, j, k*で表すと？

○ $\frac{dx^2}{dx}$ は？

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3次元空間のラグランジアンと時空間

$$\mathcal{L}(x(t), v(t), t)$$

時間の一様性 $\mathcal{L}(x(t), v(t), t)$ の中の*t*に依らない

$$\mathcal{L}(x(t+T) + X, v(t+T), t+T) = \mathcal{L}(x(t), v(t), t)$$

空間の一様性 平行移動しても同じ

$$\mathcal{L}(x(t) + X, v(t), t) = \mathcal{L}(x(t), v(t), t)$$

空間の等方性 回転しても同じ

$$\mathcal{L}(x'(t), v'(t), t) = \mathcal{L}(x(t), v(t), t) \Leftarrow \begin{cases} x'(t): x(t) を回転させる \\ v'(t): v(t) を同時に同じだけ回転させる \end{cases}$$

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時間の一様性と保存則

$$\mathcal{L}(x, v, t+T) = \mathcal{L}(x, v, t)$$

$$\mathcal{L}(x, v, t+T) = \mathcal{L}(x, v, t) + T \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t}$$

$$\begin{aligned} \mathcal{L}(x, v, t+T) &= \sum_{n=0}^{\infty} \frac{T^n}{n!} \frac{\partial^n}{\partial t^n} \mathcal{L}(x, v, t) \quad (\text{テーラー展開}) \\ &= \mathcal{L}(x, v, t) + T \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} + \frac{T^2}{2!} \frac{\partial^2 \mathcal{L}(x(t), v(t), t)}{\partial t^2} + \dots \quad (\text{無限に統合}) \end{aligned}$$

時間の一様性があると…

$$\mathcal{L}(x, v, t+T) = \mathcal{L}(x, v, t) + T \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = \mathcal{L}(x, v, t)$$

$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = 0$$

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$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = 0$$

$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t}$ の計算のため…全微分 $\frac{d \mathcal{L}(x(t), v(t), t)}{dt}$

$$\frac{d \mathcal{L}(x(t), v(t), t)}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} + \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t}$$

$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = \frac{d \mathcal{L}(x(t), v(t), t)}{dt} - \left[\frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x, v, t)}{\partial x} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x, v, t)}{\partial v} \right] = 0$$

$$\frac{d \mathcal{L}(x(t), v(t))}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

$$\begin{aligned} \frac{d \mathcal{L}(x(t), v(t))}{dt} &= \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \\ \frac{d \mathcal{L}(x(t), v(t))}{dt} &= \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \\ &\quad \text{ラグランジの運動方程式…} \\ \frac{d}{dt} \left[\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right] &= \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} \\ &= \boxed{\frac{dx(t)}{dt}} \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) + \frac{d \mathcal{L}(x(t), v(t))}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \\ &= \boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)} + \frac{d \mathcal{L}(x(t), v(t))}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \quad \text{〔公式〕} A \frac{dB}{dt} + \frac{dA}{dt} B = \frac{d}{dt}(AB) \\ &= \boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)} \\ \frac{d \mathcal{L}(x(t), v(t))}{dt} &= \frac{d}{dt} \left(\mathbf{v}(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) \end{aligned}$$

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$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)$$

保存量…

$$\begin{aligned} \frac{d\mathcal{L}(x(t), v(t))}{dt} &= \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) \\ &\Rightarrow \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) - \frac{d\mathcal{L}(x(t), v(t))}{dt} = 0 \\ &= \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t)) \right) = 0 \\ &\Rightarrow \boxed{\frac{dE}{dt} = 0} \quad \left(\leftarrow E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t)) \right) \\ &\text{Eは保存量。} \end{aligned}$$

$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t))$ は保存する

エネルギー(?)保存則

$$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t))$$

ニュートン力学の場合に、エネルギーに一致するはず！！！

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(x(t), v(t)) &= \frac{1}{2} m [v(t)]^2 - U(x(t)) \\ E &= v(t) \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} - \mathcal{L}_{\text{ニュートン}}(x(t), v(t)) \quad [\text{公式}] \frac{dv^2}{dv} = 2v \\ &\Leftrightarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} = \frac{\hat{v}}{\hat{v}(t)} \left(\frac{1}{2} m [v(t)]^2 - \frac{\nabla U(x)}{U(x)} \right) = \frac{1}{2} m \frac{\hat{v} [v(t)]^2}{\hat{v}(t)} = \frac{1}{2} m \frac{d[v(t)]^2}{d\hat{v}(t)} = m v(t) \\ &\quad \xrightarrow{\substack{\text{微分} \\ \hat{v}(t) \hat{v} - \hat{v}^2}} \xrightarrow{\substack{\text{全エネルギーに一致} \\ \hat{v} \sqrt{v^2 + U}}} \\ &= v(t) [m v(t)] - \left[\frac{1}{2} m [v(t)]^2 - U(x(t)) \right] = \frac{1}{2} m [v(t)]^2 + U(x(t)) \end{aligned}$$

$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t))$ はエネルギーであり保存する

N粒子系のエネルギー

$$E = \sum_{i=1}^N \frac{\partial \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} - \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))$$

ニュートン力学の場合に、これは全エネルギー？

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t)) &= \frac{1}{2} \sum_{i=1}^N m_i [v_i(t)]^2 - U(x_1(t), x_2(t), \dots, x_N(t)) \\ E &= \sum_{i=1}^N \frac{\partial \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} \\ &\quad - \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t)) \\ &\Leftrightarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} = \frac{1}{2} \sum_{i=1}^N m_i [v_i(t)]^2 - U(x_1(t), x_2(t), \dots, x_N(t)) \\ &= \frac{1}{2} \sum_{i=1}^N m_i \frac{\hat{v}[v_i(t)]^2}{\hat{v}(t)} = \frac{1}{2} \sum_{i=1}^N m_i \frac{d[v_i(t)]^2}{d\hat{v}(t)} = \frac{1}{2} \sum_{i=1}^N m_i \delta_{\hat{v}} \hat{v} v_i(t) = m_i v_i(t) \\ &= \sum_{i=1}^N \frac{\partial \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} - \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t)) \\ &\quad \xrightarrow{\substack{\text{全エネルギーに一致} \\ \hat{v} \sqrt{v^2 + U}}} \\ &= \frac{1}{2} m \sum_{i=1}^N [v_i(t)]^2 + U(x_1(t), x_2(t), \dots, x_N(t)) \end{aligned}$$

$E = \sum_{i=1}^N \frac{\partial \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} - \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))$

空間の一様性と保存則

$$\mathcal{L}(x(t) + X, v(t), t) = \mathcal{L}(x(t), v(t), t)$$

$$\mathcal{L}(x(t) + X, v(t), t) = \mathcal{L}(x(t), v(t), t) + X \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} \stackrel{!}{=} \mathcal{L}(x(t), v(t), t)$$

参考】 $X \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} = \frac{x(t) + X(t) - x(t)}{X(t) - x(t)} \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} + Y \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial y(t)} + Z \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial z(t)}$

$$\Rightarrow \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} = \vec{0}$$

ラグランジュの運動方程式…

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = 0$$

$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)}$ は保存する

運動量(p と書く)保存則

$$p = \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

ニュートン力学の場合に、運動量に一致するはず！！！

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(x(t), v(t)) &= \frac{1}{2} m [v(t)]^2 - U(x(t)) \\ p &= \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} \\ &\Leftrightarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} = \frac{\partial}{\partial v(t)} \frac{1}{2} m [v(t)]^2 = m v(t) \\ &\quad \xrightarrow{\substack{\text{運動量} \\ = m v(t)}} \\ p &= \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \end{aligned}$$

$p = \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$ は運動量であり保存する

N粒子系の運動量

$$p = \sum_{i=1}^N \frac{\partial \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)}$$

ニュートン力学の場合に、これは全運動量？

$$\begin{aligned} \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t)) &= \frac{1}{2} \sum_{i=1}^N m_i [v_i(t)]^2 - U(x_1(t), x_2(t), \dots, x_N(t)) \\ p &= \sum_{i=1}^N \frac{\partial \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} \\ &\Leftrightarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)} = \frac{\partial}{\partial v_i(t)} \frac{1}{2} \sum_{i=1}^N m_i [v_i(t)]^2 = \frac{1}{2} \sum_{i=1}^N m_i \frac{\partial [v_i(t)]^2}{\partial v_i(t)} = \frac{1}{2} \sum_{i=1}^N m_i \delta_{\hat{v}} \hat{v} v_i(t) \\ &\quad \xrightarrow{\substack{\text{全運動量に一致} \\ \hat{v} \sqrt{v^2 + U}}} \\ &= \sum_{i=1}^N m_i v_i(t) \end{aligned}$$

$p = \sum_{i=1}^N \frac{\partial \mathcal{L}(x_1(t), x_2(t), \dots, x_N(t), v_1(t), v_2(t), \dots, v_N(t))}{\partial v_i(t)}$