

第6回:エネルギー保存則・ 運動量保存則

必要になる基礎知識

○ $f(x(t), t)$ のとき、 $\frac{df(x(t), t)}{dt}$ を $\frac{d}{dt}, \frac{\partial}{\partial x}$ を含む式で表すと？

○ $\frac{dx^2}{dx}$ は？

○ $\mathbf{x} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ のとき $\frac{d}{d\mathbf{x}}$ を $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ と $\mathbf{i}, \mathbf{j}, \mathbf{k}$ で表すと？

○ $\frac{d\mathbf{x}^2}{d\mathbf{x}}$ は？

3次元空間のラグランジアンと時空間

$$\mathcal{L}(x(t), v(t), t)$$

時間の一様性 $\mathcal{L}(x(t), v(t), t)$ の中の t に依らない

$$\mathcal{L}(x(t+T)+X, v(t+T), t+T) = \mathcal{L}(x(t), v(t), t)$$

空間の一様性 平行移動しても同じ

$$\mathcal{L}(x(t)+X, v(t), t) = \mathcal{L}(x(t), v(t), t)$$

空間の等方性 回転しても同じ

$$\mathcal{L}(x'(t), v'(t), t) = \mathcal{L}(x(t), v(t), t) \Leftarrow \begin{cases} x'(t): x(t) \text{を回転させる} \\ v'(t): v(t) \text{を同時に同じ} \\ \text{だけ回転させる} \end{cases}$$

時間の一様性と保存則

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t+T) = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)$$

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t+T) \doteq \mathcal{L}(\mathbf{x}, \mathbf{v}, t) + T \frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial t}$$

$$\Leftarrow \mathcal{L}(\mathbf{x}, \mathbf{v}, t+T) = \sum_{n=0}^{\infty} \frac{T^n}{n!} \frac{\partial^n}{\partial t^n} \mathcal{L}(\mathbf{x}, \mathbf{v}, t) \text{ (テーラー展開)}$$

$$T \doteq 0$$



$$= \mathcal{L}(\mathbf{x}, \mathbf{v}, t) + T \frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial t} + \frac{T^2}{2!} \frac{\partial^2 \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial t^2} + \dots \text{ (無限に続く)}$$

時間の一様性があると・・・

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t+T) = \mathcal{L}(\mathbf{x}, \mathbf{v}, t) + T \frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial t} = \mathcal{L}(\mathbf{x}, \mathbf{v}, t)$$

$$\frac{\partial \mathcal{L}(\mathbf{x}(t), \mathbf{v}(t), t)}{\partial t} = 0$$

$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = 0$$

$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t}$ の計算のため... 全微分 $\frac{d\mathcal{L}(x(t), v(t), t)}{dt}$

$$\frac{d\mathcal{L}(x(t), v(t), t)}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} + \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t}$$

$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial t} = \frac{d\mathcal{L}(x(t), v(t), t)}{dt} - \left[\frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x, v, t)}{\partial x} + \frac{dv}{dt} \frac{\partial \mathcal{L}(x, v, t)}{\partial v} \right] = 0$$

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{dx(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)} + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

ラグランジュの運動方程式・・・

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) = \frac{\partial \mathcal{L}(x(t), v(t))}{\partial x(t)}$$

$$= \frac{dx(t)}{dt} \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) + \frac{dv(t)}{dt} \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

$$= \overbrace{v(t)}^A \frac{d}{dt} \left(\overbrace{\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)}}^B \right) + \frac{d \overbrace{v(t)}^A}{dt} \overbrace{\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)}}^B \leftarrow \text{【公式】 } A \frac{dB}{dt} + \frac{dA}{dt} B = \frac{d}{dt} (AB)$$

$$= \frac{d}{dt} \left(\overbrace{v(t)}^A \overbrace{\frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}}^B \right)$$

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)$$

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)$$

保存量...

$$\frac{d\mathcal{L}(x(t), v(t))}{dt} = \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right)$$

$$\Rightarrow \frac{d}{dt} \left(v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \right) - \frac{d\mathcal{L}(x(t), v(t))}{dt} = 0$$

$$\Rightarrow \frac{d}{dt} \left(\overbrace{v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t))}^E \right) = 0$$

$$\Rightarrow \frac{dE}{dt} = 0 \left(\leftarrow E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t)) \right)$$

E は保存する。 E は保存量。

$$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t)) \text{ は保存する}$$

エネルギー(?)保存則

$$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t))$$

ニュートン力学の場合に、エネルギーに一致するはず!!!

$$\mathcal{L}_{\text{ニュートン}}(x(t), v(t)) = \frac{1}{2} m [v(t)]^2 - U(x(t))$$

$$E = v(t) \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} - \mathcal{L}_{\text{ニュートン}}(x(t), v(t))$$

【公式】 $\frac{dv^2}{dv} = 2v$

$$\begin{aligned} \leftarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} &= \frac{\partial}{\partial v(t)} \left(\frac{1}{2} m [v(t)]^2 - \overbrace{U(x)}^{v(t) \text{ は いないので}} \right) = \frac{1}{2} m \underbrace{\frac{\partial ([v(t)]^2)}{\partial v(t)}}_{\text{偏微分}} = \frac{1}{2} m \underbrace{\frac{d([v(t)]^2)}{dv(t)}}_{\text{全微分}} = mv(t) \end{aligned}$$

v(t) しかない

$$= v(t) [mv(t)] - \left[\frac{1}{2} m [v(t)]^2 - U(x(t)) \right] = \overbrace{\frac{1}{2} m [v(t)]^2 + U(x(t))}^{\text{全エネルギーに一致}}$$

$$E = v(t) \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} - \mathcal{L}(x(t), v(t)) \text{ はエネルギーであり保存する}$$

N粒子系のエネルギー

$$E = \sum_{i=1}^N \mathbf{v}_i(t) \frac{\partial \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} - \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))$$

ニュートン力学の場合に、これは全エネルギー？

$$\mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t)) = \frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2 - U(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t))$$

$$E = \sum_{i=1}^N \mathbf{v}_i(t) \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} - \mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))$$

$$\frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} = \frac{\partial \left(\frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2 - U(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t)) \right)}{\partial \mathbf{v}_i(t)}$$

$$= \frac{1}{2} \sum_{k=1}^N m_k \frac{\partial [\mathbf{v}_k(t)]^2}{\partial \mathbf{v}_i(t)} = \frac{1}{2} \sum_{k=1}^N m_k \frac{d[\mathbf{v}_k(t)]^2}{d\mathbf{v}_i(t)} = \frac{1}{2} \sum_{k=1}^N m_k \delta_{ik} 2\mathbf{v}_k(t) = m_i \mathbf{v}_i(t)$$

$$= \sum_{i=1}^N \mathbf{v}_i(t) m_i \mathbf{v}_i(t) - \left[\frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2 - U(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t)) \right]$$

全エネルギーに一致

$$= \frac{1}{2} m \sum_{k=1}^N [\mathbf{v}_k(t)]^2 + U(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t))$$

【公式】 $\frac{d[\mathbf{v}_k(t)]^2}{d\mathbf{v}_i(t)} = \begin{cases} k=i \dots \frac{d[\mathbf{v}_i(t)]^2}{d\mathbf{v}_i(t)} = 2\mathbf{v}_k(t) \\ k \neq i \dots 0 \end{cases}$

$= 2\delta_{ik} \mathbf{v}_k(t) \Leftarrow \delta_{ik} = \begin{cases} k=i \dots 1 \\ k \neq i \dots 0 \end{cases}$

$$E = \sum_{i=1}^N \mathbf{v}_i(t) \frac{\partial \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} - \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))$$

空間の一様性と保存則

$$\mathcal{L}(x(t) + X, v(t), t) = \mathcal{L}(x(t), v(t), t)$$

$$\mathcal{L}(x(t) + X, v(t), t) \doteq \mathcal{L}(x(t), v(t), t) + X \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} \stackrel{\downarrow}{=} \mathcal{L}(x(t), v(t), t)$$

【参考】 $X \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} \stackrel{\substack{x(t)=(x(t), y(t), z(t)) \\ X=(X, Y, Z)}}{=} X \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} + Y \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial y(t)} + Z \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial z(t)}$

$$\Rightarrow \overbrace{\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)}}^{\text{ベクトル}} = \underbrace{\vec{0}}_{\text{零ベクトル}}$$

ラグランジュの運動方程式・・・

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = \frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial x(t)} = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \right) = 0$$

$$\frac{\partial \mathcal{L}(x(t), v(t), t)}{\partial v(t)} \text{ は保存する}$$

運動量 (p と書く) 保存則

$$p = \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)}$$

ニュートン力学の場合に、運動量に一致するはず！！！！

$$\mathcal{L}_{\text{ニュートン}}(x(t), v(t)) = \frac{1}{2}m[v(t)]^2 - U(x(t))$$

$$p = \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)}$$

$$\leftarrow \frac{\partial \mathcal{L}_{\text{ニュートン}}(x(t), v(t))}{\partial v(t)} = \frac{\partial \frac{1}{2}m[v(t)]^2}{\partial v(t)} = mv(t)$$

$$\underbrace{= mv(t)}_{\text{運動量}}$$

$$p = \frac{\partial \mathcal{L}(x(t), v(t))}{\partial v(t)} \text{ は運動量であり保存する}$$

N粒子系の運動量

$$\mathbf{p} = \sum_{i=1}^N \frac{\partial \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)}$$

ニュートン力学の場合に、これは全運動量？

$$\mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t)) = \frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2 - U(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t))$$

$$\mathbf{p} = \sum_{i=1}^N \frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)}$$

$$\frac{\partial \mathcal{L}_{\text{ニュートン}}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)} = \frac{\partial \frac{1}{2} \sum_{k=1}^N m_k [\mathbf{v}_k(t)]^2}{\partial \mathbf{v}_i(t)} = \frac{1}{2} \sum_{k=1}^N m_k \frac{\partial [\mathbf{v}_k(t)]^2}{\partial \mathbf{v}_i(t)}$$

$$= \frac{1}{2} \sum_{k=1}^N m_k \delta_{ik} 2\mathbf{v}_k(t) = m_i \mathbf{v}_i(t)$$

全運動量に一致

$$= \sum_{i=1}^N m_i \mathbf{v}_i(t)$$

$$\mathbf{p} = \sum_{i=1}^N \frac{\partial \mathcal{L}(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_N(t), \mathbf{v}_1(t), \mathbf{v}_2(t), \dots, \mathbf{v}_N(t))}{\partial \mathbf{v}_i(t)}$$