第5回：ニュートンの運動方程式

必要になる基礎知識

⊙$\boldsymbol{a}=\left(\boldsymbol{a}_x,\boldsymbol{a}_y,\boldsymbol{a}_z\right)$の時、$\boldsymbol{a}\cdot\boldsymbol{a}\left(=\boldsymbol{a}^2\text{と書く}\right)$は？

⊙直交座標(x,y,z)とするとき極座標(r,θ,ϕ)を用いてx,y,zを表すと？

⊙直交座標(x,y,z)とするとき速度$\left(v_x,v_y,v_z\right)$をx,y,zで表すと？

⊙極座標(r,θ,ϕ)とするとき速度$\left(v_r,v_\theta,v_\phi\right)$を$r,\theta,\phi$で表すと？

力のある場合とラグランジュの方法

力がないと？？？　空間の一様性　$\dfrac{\partial \mathcal{L}(x(t),v(t))}{\partial x(t)}=0$

空間の等方性　$\mathcal{L}(x(t),v(t))=\mathcal{L}(x(t),-v(t))$

$$\boxed{\mathcal{L}(x(t),v(t))=\frac{1}{2}m\left[v(t)\right]^2}$$

力があると？？？　空間の一様性が成立しない

$$\frac{\partial \mathcal{L}(x(t),v(t))}{\partial x(t)}\neq 0$$

空間の等方性が成立しない

$$\mathcal{L}(x(t),v(t))\neq\mathcal{L}(x(t),-v(t))$$

空間が一様でないとき

力があると？？？　空間の一様性が成立しない

$$\frac{\partial \mathcal{L}\left(x(t), v(t)\right)}{\partial x(t)} \neq 0$$

力が無いとき　$\mathcal{L}\left(x(t), v(t)\right) = \frac{1}{2} m \left[v(t)\right]^2$

力があるとき　$\mathcal{L}\left(x(t), v(t)\right) = \frac{1}{2} m \left[v(t)\right]^2 - U\left(x(t)\right)$

$$\because \frac{\partial \mathcal{L}\left(x(t), v(t)\right)}{\partial x(t)} = \frac{\partial \left\{\frac{1}{2} m \left[v(t)\right]^2 - U\left(x(t)\right)\right\}}{\partial x(t)} = -\frac{\partial U\left(x(t)\right)}{\partial x(t)} \neq 0$$

$\Rightarrow$ 空間が一様でない

$\Rightarrow$ 力があるとき

$$\boxed{\mathcal{L}\left(x(t), v(t)\right) = \frac{1}{2} m \left[v(t)\right]^2 - U\left(x(t)\right)}$$

ラグランジュの方法と保存力

$$\boxed{\mathcal{L}\left(x(t),v(t)\right)=\frac{1}{2}m\left[v(t)\right]^2-U\left(x(t)\right)}$$

ラグランジュの運動方程式

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}\left(x(t),v(t)\right)}{\partial v(t)}\right)=\frac{\partial\mathcal{L}\left(x(t),v(t)\right)}{\partial x(t)}$$

$$\frac{\partial\mathcal{L}\left(x(t),v(t)\right)}{\partial v(t)}=\frac{\partial\left(\frac{1}{2}m\left[v(t)\right]^2-U\left(x(t)\right)\right)}{\partial v(t)}=mv(t)$$

$$\frac{\partial\mathcal{L}\left(x(t),v(t)\right)}{\partial x(t)}=\frac{\partial\left(\frac{1}{2}m\left[v(t)\right]^2-U\left(x(t)\right)\right)}{\partial x(t)}=-\frac{\partial U\left(x(t)\right)}{\partial x(t)}$$

$$\boxed{\frac{d}{dt}\left(mv(t)\right)=-\frac{\partial U\left(x(t)\right)}{\partial x(t)}}$$

$$\Rightarrow$$

$$\boxed{\overbrace{m\frac{dv(t)}{dt}}^{\text{ニュートンの力}}=\overbrace{-\frac{\partial U\left(x(t)\right)}{\partial x(t)}}^{\text{保存力}}}$$

3次元ベクトル表記:直交座標

$$\boxed{\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\boldsymbol{v}(t)\right]^2-U\left(\boldsymbol{x}(t)\right)\quad\left(\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}\right)}$$

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\boldsymbol{v}(t)\right]^2-U\left(\boldsymbol{x}(t)\right)\quad\left(\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}\right)$$

$$\boldsymbol{x}(t)=\left(x(t),y(t),z(t)\right)$$

$$\boldsymbol{v}(t)=\left(\boldsymbol{v}_x(t),\boldsymbol{v}_y(t),\boldsymbol{v}_z(t)\right)$$

$$\left[\boldsymbol{v}(t)\right]^2=\boldsymbol{v}_x(t)\boldsymbol{v}_x(t)+\boldsymbol{v}_y(t)\boldsymbol{v}_y(t)+\boldsymbol{v}_z(t)\boldsymbol{v}_z(t)$$

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}=\left(\frac{dx(t)}{dt},\frac{dy(t)}{dt},\frac{dz(t)}{dt}\right)$$

$$=\frac{1}{2}m\left(\left[\boldsymbol{v}_x(t)\right]^2+\left[\boldsymbol{v}_y(t)\right]^2+\left[\boldsymbol{v}_z(t)\right]^2\right)-U\left(x(t),y(t),z(t)\right)$$

$$=\frac{1}{2}m\left[\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2\right]-U\left(x(t),y(t),z(t)\right)$$

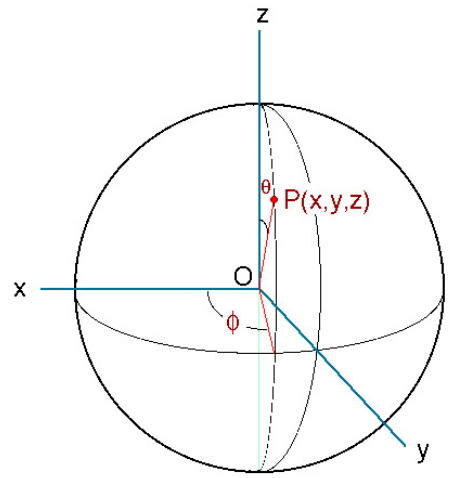
$$\boxed{\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2\right]-U\left(x(t),y(t),z(t)\right)}$$

3次元ベクトル表記：極座標

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\boldsymbol{v}(t)\right]^2-U\left(\boldsymbol{x}(t)\right)\quad\left(\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}\right)$$

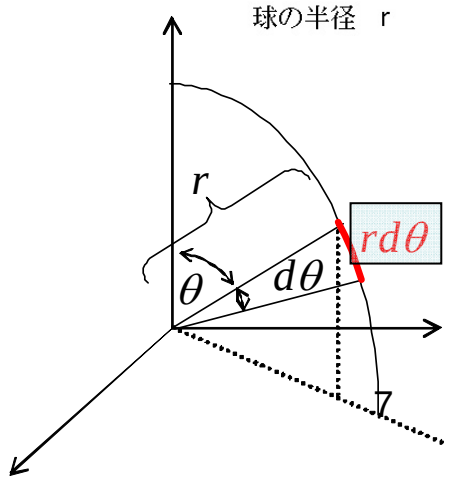
位置は・・・ $\boldsymbol{x}(t)=(r,\theta,\phi)$

速度は・・・ $\boldsymbol{v}(t)=\left(\frac{dr}{dt},\cancel{\frac{d\theta}{dt}},\cancel{\frac{d\phi}{dt}}\right)$



速度を計算する・・・

$$\frac{dx}{dt}=\frac{\text{微小移動距離}}{\text{微小経過時間}}$$ と考える

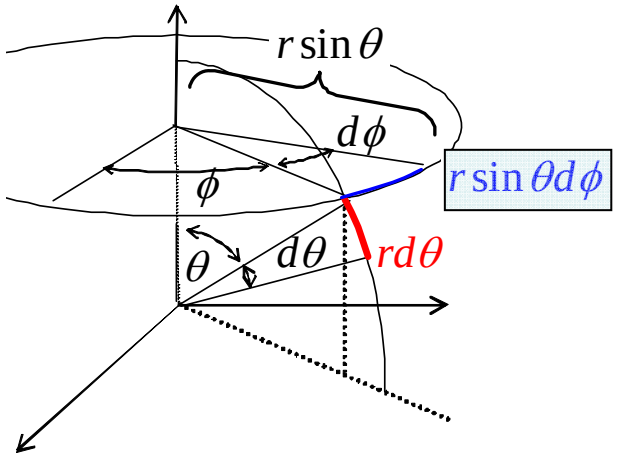


円弧の移動距離は・・・ $\boxed{rd\theta}$

$$\frac{\text{微小移動距離}}{\text{微小経過時間}}=\frac{rd\theta}{dt}\quad\Rightarrow\quad\boxed{v_\theta=r\frac{d\theta}{dt}}$$

$$v(t)=\left(\frac{dr}{dt},\cancel{\frac{d\theta}{dt}},\cancel{\frac{d\phi}{dt}}\right) \Rightarrow v(t)=\left(\frac{dr}{dt},r\frac{d\theta}{dt},\cancel{\frac{d\phi}{dt}}\right)$$

もうひとつの円弧の移動距離は･･･



$$\boxed{r\sin\theta d\phi}$$

$$\frac{\text{微小移動距離}}{\text{微小経過時間}}=\frac{r\sin\theta d\phi}{dt}$$

$$\Rightarrow \boxed{v_\phi = r\sin\theta\frac{d\phi}{dt}}$$

$$v(t)=\left(\frac{dr(t)}{dt},r\frac{d\theta(t)}{dt},r\sin\theta\frac{d\phi(t)}{dt}\right)$$

$$\boxed{\boldsymbol{v}(t)=\left(\frac{dr(t)}{dt},r(t)\frac{d\theta(t)}{dt},r(t)\sin\theta(t)\frac{d\phi(t)}{dt}\right)}$$

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\boldsymbol{v}(t)\right]^2-U\left(\boldsymbol{x}(t)\right)\quad\left(\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}\right)$$

$$\boldsymbol{x}(t)=\left(r(t),\theta(t),\phi(t)\right)$$

$$\boldsymbol{v}(t)=\left(\boldsymbol{v}_r(t),\boldsymbol{v}_\theta(t),\boldsymbol{v}_\phi(t)\right)$$

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}=\left(\frac{dr}{dt},r\frac{d\theta}{dt},r\sin\theta\frac{d\phi}{dt}\right)$$

$$=\frac{1}{2}m\left(\left[\boldsymbol{v}_x(t)\right]^2+\left[\boldsymbol{v}_y(t)\right]^2+\left[\boldsymbol{v}_z(t)\right]^2\right)-U\left(r(t),\theta(t),\phi(t)\right)$$

$$=\frac{1}{2}m\left[\left(\frac{dr(t)}{dt}\right)^2+\left(r\frac{d\theta(t)}{dt}\right)^2+\left(r\sin\theta\frac{d\phi(t)}{dt}\right)^2\right]-U\left(r(t),\theta(t),\phi(t)\right)$$

$$\boxed{\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\left(\frac{dr(t)}{dt}\right)^2+\left(r(t)\frac{d\theta(t)}{dt}\right)^2+\left(r(t)\sin\theta(t)\frac{d\phi(t)}{dt}\right)^2\right]-U\left(r(t),\theta(t),\phi(t)\right)}$$

3次元ベクトル表記（まとめ）

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\boldsymbol{v}(t)\right]^2-U\left(\boldsymbol{x}(t)\right)\quad\left(\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}\right)$$

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2\right]-U\left(x(t),y(t),z(t)\right)$$

$$\boldsymbol{x}(t)=\left(x(t),y(t),z(t)\right)$$

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}=\left(\frac{dx(t)}{dt},\frac{dy(t)}{dt},\frac{dz(t)}{dt}\right)$$

$$\mathcal{L}\left(\boldsymbol{x}(t),\boldsymbol{v}(t)\right)=\frac{1}{2}m\left[\left(\frac{dr(t)}{dt}\right)^2+\left(r(t)\frac{d\theta(t)}{dt}\right)^2+\left(r(t)\sin\theta(t)\frac{d\phi(t)}{dt}\right)^2\right]-U\left(r(t),\theta(t),\phi(t)\right)$$

$$\boldsymbol{x}(t)=\left(r(t),\theta(t),\phi(t)\right)$$

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{x}(t)}{dt}=\left(\frac{dr}{dt},r\frac{d\theta}{dt},r\sin\theta\frac{d\phi}{dt}\right)$$