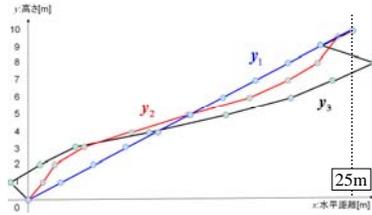


第3回(続き):ラグランジュの運動方程式一例

1

最小曲線の原理:x積分



2点間の最小距離は直線である

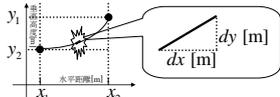
$$S(y, \dot{y}) = \int_{x_1(=0\text{m})}^{x_2(=25\text{m})} \mathcal{L}(y(x), \dot{y}(x)) dx \quad \left(\dot{y}(x) = \frac{dy(x)}{dx} \right)$$

2

曲線の長さ

微小2点間の曲線の距離は直線である

身近な例:地球表面上の微小2点間は直線である



微小2点間の曲線長さは $\sqrt{(dx)^2 + (dy)^2}$ (3平方の定理)

$$\text{曲線の全長} = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2} dx$$

$$S(y, \dot{y}) = \int_{x_1}^{x_2} \mathcal{L}(y(x), \dot{y}(x)) dx = \int_{x_1}^{x_2} \sqrt{1 + (\dot{y}(x))^2} dx$$

$$\mathcal{L}(y(x), \dot{y}(x)) = \sqrt{1 + (\dot{y}(x))^2}$$

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曲線の長さからラグランジュ方程式

$$\mathcal{L}(y(x), \dot{y}(x)) = \sqrt{1 + (\dot{y}(x))^2}$$

ラグランジュ方程式

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)}$$

$$\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} = \frac{\partial}{\partial \dot{y}(x)} \sqrt{1 + (\dot{y}(x))^2} = \frac{1}{2} \frac{2\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} = \frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}}$$

$$\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)} = \frac{\partial}{\partial y(x)} \sqrt{1 + (\dot{y}(x))^2} = 0$$

ラグランジュ方程式は...

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = 0 \Rightarrow \frac{d}{dx} \left(\frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} \right) = 0 \Rightarrow \frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} = C (= \text{定数})$$

$$\Rightarrow \dot{y}(x) = a (= \text{新たな定数}) \Rightarrow \frac{dy(x)}{dx} = a \Rightarrow y(x) = \frac{a}{1} x + b$$

直線の方程式

2点間の最小距離を与える曲線は直線になる

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最小曲線の原理:t積分

標準形(t積分)での考察 $S(x, \dot{x}) = \int_t^t \mathcal{L}(x(t), \dot{x}(t)) dt$

$$\text{曲線の全長} = \int \sqrt{(dx)^2 + (dy)^2}$$

簡易形(x積分)

$$S(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\stackrel{y=\dot{y}x}{=} \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2} dx \Rightarrow \mathcal{L} = \sqrt{1 + \dot{y}^2}$$

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)}$$

標準形(t積分)

$$S(x_1, x_2, t_1, t_2) = \int_{t_1}^{t_2} \sqrt{(dx)^2 + (dy)^2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\stackrel{x=\dot{x}t}{=} \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} dt \Rightarrow \mathcal{L} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{x}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{y}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial y(t)}$$

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最小曲線は直線:t積分

$$\mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t)) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{x}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{y}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial y(t)}$$

左辺: $\frac{\partial \mathcal{L}}{\partial \dot{x}(t)} = \frac{\partial \sqrt{\dot{x}^2 + \dot{y}^2}}{\partial \dot{x}(t)} = \frac{\partial (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}}{\partial \dot{x}(t)} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}} 2\dot{x}(t) = \frac{\dot{x}(t)}{\sqrt{\dot{x}^2 + \dot{y}^2}}$

右辺: $\frac{\partial \mathcal{L}}{\partial x(t)} = \frac{\partial \sqrt{\dot{x}^2 + \dot{y}^2}}{\partial x(t)} = 0$

左辺: $\frac{\partial \mathcal{L}}{\partial \dot{y}(t)} = \frac{\partial \sqrt{\dot{x}^2 + \dot{y}^2}}{\partial \dot{y}(t)} = \frac{\partial (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}}{\partial \dot{y}(t)} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}} 2\dot{y}(t) = \frac{\dot{y}(t)}{\sqrt{\dot{x}^2 + \dot{y}^2}}$

右辺: $\frac{\partial \mathcal{L}}{\partial y(t)} = \frac{\partial \sqrt{\dot{x}^2 + \dot{y}^2}}{\partial y(t)} = 0$

$$\frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0 \quad \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0$$

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$\frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 \quad \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0$

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 &\Rightarrow \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = C (= \text{定数}) \\ \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 &\Rightarrow \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = D (= \text{定数}) \end{aligned} \right\} \Rightarrow \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \bigg/ \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = \frac{D}{C}$$

$$\Rightarrow \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{D}{C} (= a \text{ とする}) \Rightarrow \dot{y}(t) = a\dot{x}(t) \Rightarrow \frac{dy(t)}{dt} = a \frac{dx(t)}{dt}$$

$$\Rightarrow \frac{d}{dt} [y(t) - ax(t)] = 0$$

$$\Rightarrow y(t) - ax(t) = b (= \text{定数})$$

$$\Rightarrow y(t) = a x(t) + b$$

定数 定数
 $y(t) = a x(t) + b$
 直線の方程式

2点間の最小距離を与える曲線は直線になる 7

最小曲線と相対性理論

曲線の全長 = $\int \sqrt{(dx)^2 + (dy)^2}$

"曲線"の全長 = $\int \sqrt{(cdt)^2 - [(dx)^2 + (dy)^2]}$ ($c = \text{光速} \approx \text{秒速} 30 \text{万キロ}$)

2点間の光の軌跡 = 直線 (無重力) 2点間の光の軌跡 = 曲線 (超強力重力 = ブラックホール)

最速は光の速度 (相対性理論)

$$S = \int \sqrt{(cdt)^2 - [(dx)^2 + (dy)^2]}$$

$$\sqrt{(cdt)^2 - [(dx)^2 + (dy)^2]} = \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} (dt) = \sqrt{c^2 - v^2} dt$$

物理的条件 = $\sqrt{\text{内}}$ は正か零 $v = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2$

$$c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] \geq 0 \Rightarrow c^2 - (v_x^2 + v_y^2) \geq 0 \Rightarrow c^2 - v^2 \geq 0 \Rightarrow v \leq c$$

速度は光の速度が最大である

$$S = \int \sqrt{c^2 - [(dx)^2 + (dy)^2]} = \int \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

$$S = \int \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

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$$S = \int_{t_1}^{t_2} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

x, y は...

光の軌跡

無重力

無重力・・・直線になる

$\Rightarrow y = ax + b$

光の軌跡

超強力重力

超強力重力 (ブラックホール)

・・・光は曲げられる

\Rightarrow 例えば、円軌道???

$\Rightarrow x = r \cos \theta, y = r \sin \theta$

$x = x(r, \theta)$
 $y = y(r, \theta) = r \sin \theta$

\Downarrow $r \rightarrow X$
 $\theta \rightarrow Y$ X, Y にして一般化

$$x = x(X, Y), y = y(X, Y)$$

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$$x = x(X, Y), y = y(X, Y)$$

$$S = \int \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt \Rightarrow \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \text{を} \frac{dX}{dt}, \frac{dY}{dt} \text{で表す}$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial X} \frac{dX}{dt} + \frac{\partial x}{\partial Y} \frac{dY}{dt}$$

$$\left(\frac{dx}{dt} \right)^2 = \left(\frac{\partial x}{\partial X} \frac{dX}{dt} + \frac{\partial x}{\partial Y} \frac{dY}{dt} \right)^2 = \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dX}{dt} \frac{\partial x}{\partial X} + 2 \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dY}{dt} \frac{\partial x}{\partial Y} + \frac{\partial x}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \frac{\partial x}{\partial Y}$$

$$\left(\frac{dy}{dt} \right)^2 = \left(\frac{\partial y}{\partial X} \frac{dX}{dt} + \frac{\partial y}{\partial Y} \frac{dY}{dt} \right)^2 = \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dX}{dt} \frac{\partial y}{\partial X} + 2 \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dY}{dt} \frac{\partial y}{\partial Y} + \frac{\partial y}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \frac{\partial y}{\partial Y}$$

$$\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dX}{dt} \frac{\partial x}{\partial X} + 2 \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dY}{dt} \frac{\partial x}{\partial Y} + \frac{\partial x}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dX}{dt} \frac{\partial y}{\partial X} + 2 \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dY}{dt} \frac{\partial y}{\partial Y} + \frac{\partial y}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \frac{\partial y}{\partial Y}$$

$$= \left(\frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} \right) \frac{dX}{dt} \frac{dX}{dt} + 2 \left(\frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \right) \frac{dX}{dt} \frac{dY}{dt} + \left(\frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \right) \frac{dY}{dt} \frac{dY}{dt}$$

$$= \left(\frac{dx}{dt} \frac{dX}{dt} \right) \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \left(\frac{dx}{dt} \frac{dY}{dt} \right) \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \left(\frac{dy}{dt} \frac{dX}{dt} \right) \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} + \left(\frac{dy}{dt} \frac{dY}{dt} \right) \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y}$$

$$= \frac{dX^T}{dt} g(X) \frac{dX}{dt} = g(X) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \frac{\partial y}{\partial X} \\ \frac{\partial x}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial X} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} \\ \frac{\partial x}{\partial Y} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix} \rightarrow g^T(X) = g(X)$$

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$$\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = \frac{dX^T}{dt} g \frac{dX}{dt}$$

$$S = \int \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt \Rightarrow \mathcal{L} = \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} = \sqrt{c^2 - \frac{dX^T}{dt} g \frac{dX}{dt}}$$

← ラグランジュ方程式 $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^a} \right) = \frac{\partial \mathcal{L}}{\partial X^a}$

$$\Rightarrow \frac{d}{dt} \frac{1}{2} \frac{2 \left(-g \frac{dX}{dt} \right)_a}{\sqrt{c^2 - \frac{dX^T}{dt} g \frac{dX}{dt}}} = \frac{1}{2} \frac{dX^T}{dt} \frac{\partial g}{\partial X^a} \frac{dX}{dt} = \frac{d}{dt} \frac{2 \left(g \frac{dX}{dt} \right)_a}{\sqrt{c^2 - \frac{dX^T}{dt} g \frac{dX}{dt}}} = \frac{dX^T}{dt} \frac{\partial g}{\partial X^a} \frac{dX}{dt}$$

$$\Rightarrow 2 \frac{d}{dt} \frac{1}{\sqrt{c^2 - \frac{dX^T}{dt} g \frac{dX}{dt}}} \left(g \frac{dX}{dt} \right)_a = \frac{dX^T}{dt} \frac{\partial g}{\partial X^a} \frac{dX}{dt}$$

$$\Rightarrow 2 \frac{d}{cd\tau} \left(g \frac{dX}{cd\tau} \right)_a = \frac{dX^T}{cd\tau} \frac{\partial g}{\partial X^a} \frac{dX}{cd\tau} \leftarrow cd\tau = \sqrt{c^2 - \frac{dX^T}{dt} g \frac{dX}{dt}}$$

$$\frac{d}{cd\tau} \left(g \frac{dX}{cd\tau} \right)_a = \frac{1}{2} \frac{dX^T}{cd\tau} \frac{\partial g}{\partial X^a} \frac{dX}{cd\tau}$$

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