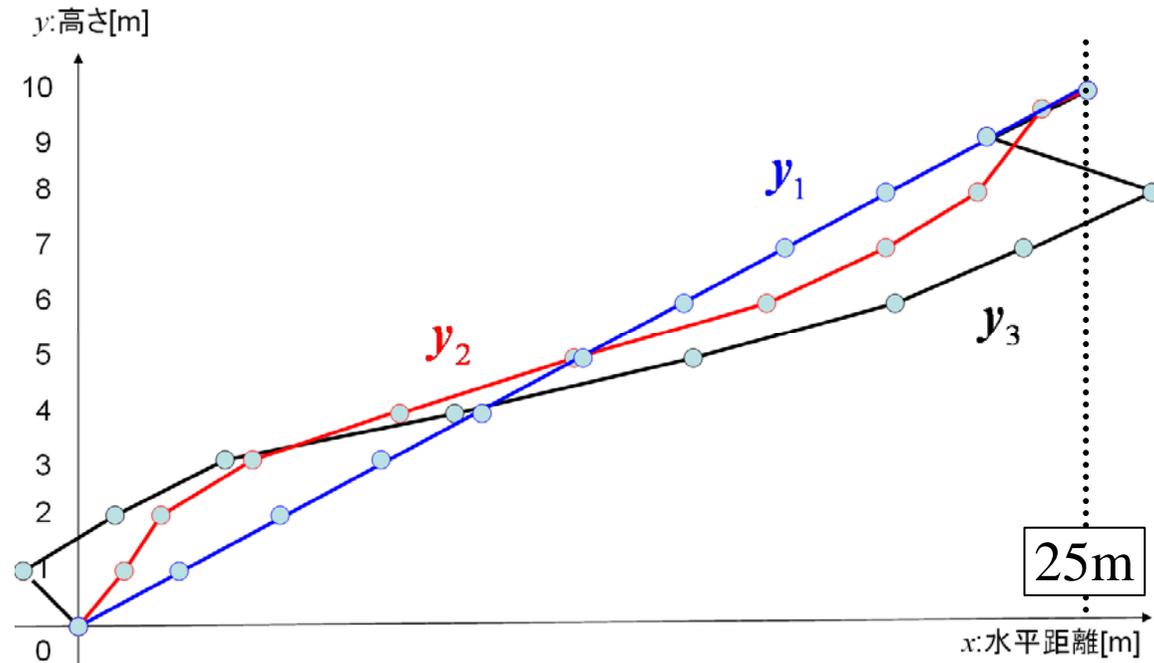


第3回(続き):ラグラン ジュの運動方程式-例

最小曲線の原理:x積分



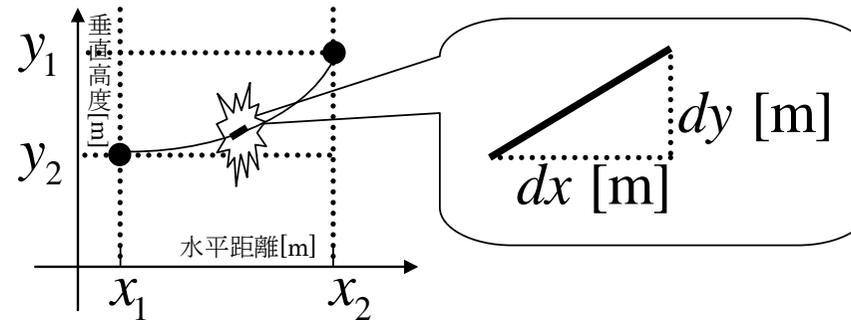
2点間の最小距離は直線である

$$S(y, \dot{y}) = \int_{x_i (=0m)}^{x_f (=25m)} \overbrace{\mathcal{L}(y(x), \dot{y}(x))}^{\text{曲線の長さ}} dx \left(\dot{y}(x) = \frac{dy(x)}{dx} \right)$$

曲線の長さ

微小2点間の曲線の距離は直線である

身近な例: 地球表面上の微小2点間は直線である



微小2点間の曲線長さ = $\sqrt{(dx)^2 + (dy)^2}$ (3平方の定理)

$$\text{曲線の全長} = \int_{x_i}^{x_f} \sqrt{(dx)^2 + (dy)^2} = \int_{x_i}^{x_f} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x_i}^{x_f} \sqrt{1 + \dot{y}^2} dx$$

$$S(y, \dot{y}) = \int_{x_i}^{x_f} \overbrace{\mathcal{L}(y(x), \dot{y}(x))}^{\text{曲線の長さ}} dx = \int_{x_i}^{x_f} \sqrt{1 + (\dot{y}(x))^2} dx$$

⇓

$$\mathcal{L}(y(x), \dot{y}(x)) = \sqrt{1 + (\dot{y}(x))^2}$$

曲線の長さとのラグランジュ方程式

$$\mathcal{L}(y(x), \dot{y}(x)) = \sqrt{1 + (\dot{y}(x))^2}$$

ラグランジュ方程式

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)}$$

$$\begin{aligned} \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} &= \frac{\partial}{\partial \dot{y}(x)} \sqrt{1 + (\dot{y}(x))^2} = \frac{1}{2} \frac{2\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} = \frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} \\ \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)} &= \frac{\partial}{\partial y(x)} \sqrt{1 + (\dot{y}(x))^2} = 0 \end{aligned}$$

ラグランジュ方程式は...

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = 0 \Rightarrow \frac{d}{dx} \left(\frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} \right) = 0 \Rightarrow \frac{\dot{y}(x)}{\sqrt{1 + (\dot{y}(x))^2}} = C (= \text{定数})$$

$$\Rightarrow \dot{y}(x) = a (= \text{新たな定数}) \Rightarrow \frac{dy(x)}{dx} = a \Rightarrow y(x) = \overset{\text{定数}}{a} x + \overset{\text{定数}}{b}$$

直線の方程式

2点間の最小距離を与える曲線は直線になる

最小曲線の原理:t積分

標準形(t積分)での考察 $S(x, \dot{x}) = \int_{t_i}^{t_f} \mathcal{L} \left(x(t), \underbrace{\frac{dx(t)}{dt}}_{\dot{x}(t)} \right) dt$

$$\text{曲線の全長} = \int \sqrt{(dx)^2 + (dy)^2}$$

簡易形(x積分)

$$S[y, \dot{y}] = \int_{x_i}^{x_f} \sqrt{(dx)^2 + (dy)^2} = \int_{x_i}^{x_f} \sqrt{1 + \left(\frac{dy(x)}{dx} \right)^2} dx$$

$$\stackrel{\dot{y} = \frac{dy}{dx}}{=} \int_{x_i}^{x_f} \sqrt{1 + \dot{y}^2} dx \Rightarrow \mathcal{L} = \sqrt{1 + \dot{y}^2(x)}$$

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial \dot{y}(x)} \right) = \frac{\partial \mathcal{L}(y(x), \dot{y}(x))}{\partial y(x)}$$

標準形(t積分)

$$S[x, y, \dot{x}, \dot{y}] = \int_{t_i}^{t_f} \sqrt{(dx)^2 + (dy)^2} = \int_{t_i}^{t_f} \sqrt{\left(\frac{dx(t)}{dt} \right)^2 + \left(\frac{dy(t)}{dt} \right)^2} dt$$

$$\stackrel{\substack{\dot{x} = \frac{dx(t)}{dt} \\ \dot{y} = \frac{dy(t)}{dt}}}{=} \int_{t_i}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2} dt \Rightarrow \mathcal{L} = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{x}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{y}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial y(t)}$$

最小曲線は直線:t積分

$$\mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t)) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{x}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial x(t)}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial \dot{y}(t)} \right) = \frac{\partial \mathcal{L}(x(t), \dot{x}(t), y(t), \dot{y}(t))}{\partial y(t)}$$

左辺: $\frac{\partial \mathcal{L}}{\partial \dot{x}(t)} = \frac{\partial \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{\partial \dot{x}(t)} = \frac{\partial (\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{1}{2}}}{\partial \dot{x}(t)} = \frac{1}{2} (\dot{x}(t)^2 + \dot{y}(t)^2)^{-\frac{1}{2}} 2\dot{x}(t) = \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}$

右辺: $\frac{\partial \mathcal{L}}{\partial x(t)} = \frac{\partial \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{\partial x(t)} = 0$

左辺: $\frac{\partial \mathcal{L}}{\partial \dot{y}(t)} = \frac{\partial \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{\partial \dot{y}(t)} = \frac{\partial (\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{1}{2}}}{\partial \dot{y}(t)} = \frac{1}{2} (\dot{x}(t)^2 + \dot{y}(t)^2)^{-\frac{1}{2}} 2\dot{y}(t) = \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}$

右辺: $\frac{\partial \mathcal{L}}{\partial y(t)} = \frac{\partial \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{\partial y(t)} = 0$

$$\frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 \quad \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0$$

$$\frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 \quad \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0$$

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 &\Rightarrow \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = C (= \text{定数}) \\ \frac{d}{dt} \left(\frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \right) = 0 &\Rightarrow \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = D (= \text{定数}) \end{aligned} \right\} \Rightarrow \frac{\dot{y}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} \bigg/ \frac{\dot{x}(t)}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}} = \frac{D}{C}$$

$$\Rightarrow \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{D}{C} = a (\text{とする}) \Rightarrow \dot{y}(t) = a\dot{x}(t) \Rightarrow \frac{dy(t)}{dt} = a \frac{dx(t)}{dt}$$

$$\Rightarrow \frac{d}{dt} [y(t) - ax(t)] = 0$$

$$\Rightarrow y(t) - ax(t) = b (= \text{定数})$$

$$\Rightarrow y(t) = \overset{\text{定数}}{\hat{a}} x(t) + \overset{\text{定数}}{\hat{b}}$$

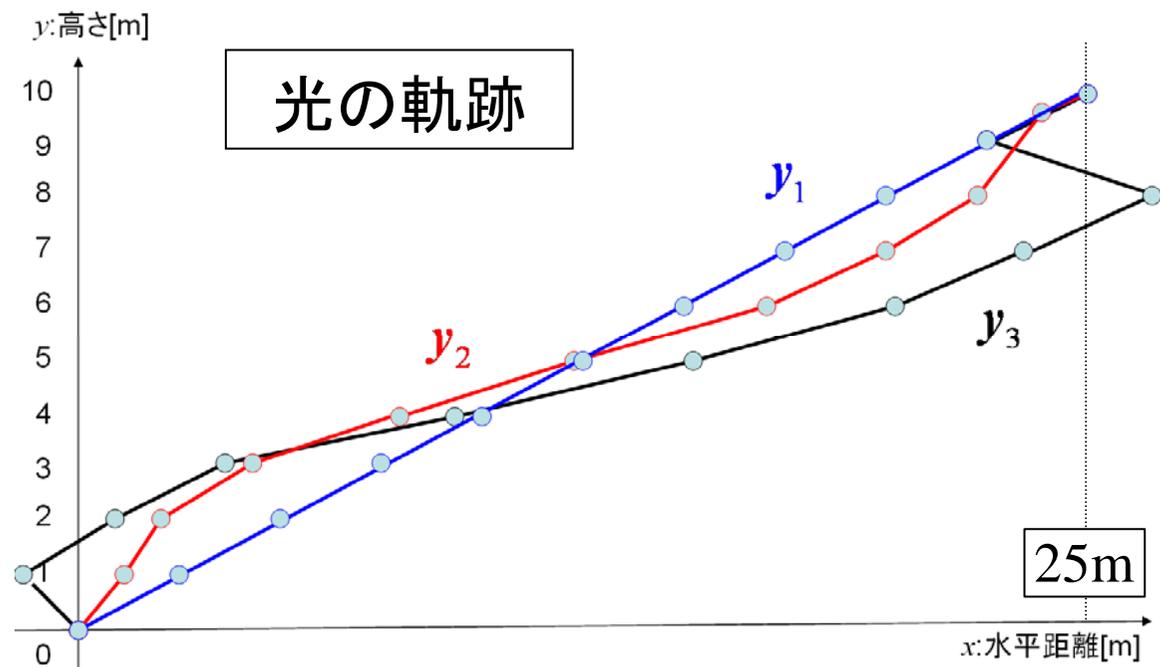
直線の方程式

2点間の最小距離を与える曲線は直線になる

最小曲線と相対性理論

$$\text{曲線の全長} = \int \sqrt{(dx)^2 + (dy)^2}$$

$$\text{"曲線"の全長} = \int \sqrt{(cdt)^2 - [(dx)^2 + (dy)^2]} \quad (c = \text{光速} \div \text{秒速30万キロ})$$



2点間の光の軌跡＝直線（無重力）

2点間の光の軌跡＝曲線（超強力重力、
＝ブラックホール）

最速は光の速度(相対性理論)

$$S = \int \sqrt{(cdt)^2 - \left[(dx)^2 + (dy)^2 \right]}$$

$$\sqrt{(cdt)^2 - \left[(dx)^2 + (dy)^2 \right]} = \sqrt{\left(c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] \right) (dt)^2} = \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

物理的条件 = $\sqrt{\quad}$ 内は正か零 $v_x^2 = \left(\frac{dx}{dt} \right)^2, v_y^2 = \left(\frac{dy}{dt} \right)^2$

$$c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] \geq 0 \Rightarrow c^2 - (v_x^2 + v_y^2) \geq 0 \Rightarrow c^2 - v^2 \geq 0 \Rightarrow v^2 \leq c^2$$

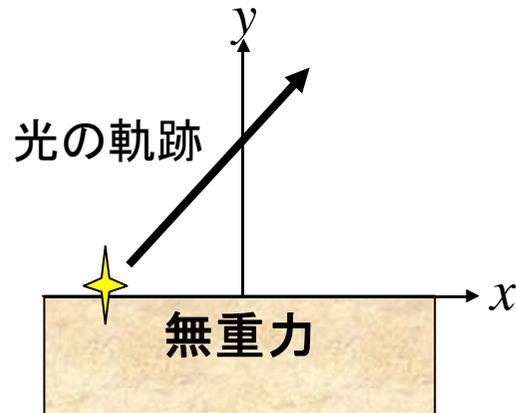
速度は光の速度が最大である

$$S = \int \sqrt{(cdt)^2 - \left[(dx)^2 + (dy)^2 \right]} = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

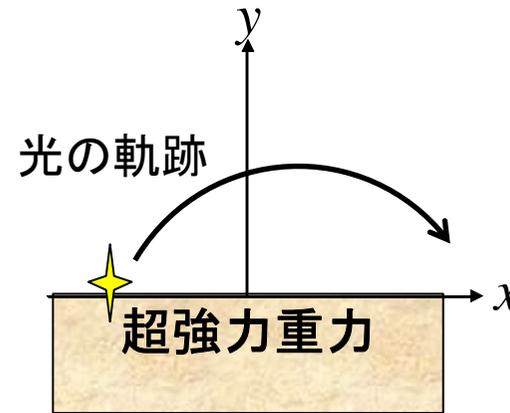
$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

x, y は・・・



無重力・・・直線になる

$$\Rightarrow y = ax + b$$



超強力重力（ブラックホール）

・・・光は曲げられる

\Rightarrow 例えば、円軌道???

$$\Rightarrow x = r \cos \theta, y = r \sin \theta$$

$$x = x(r, \theta) = r \cos \theta$$

$$y = y(r, \theta) = r \sin \theta$$

$$\Downarrow \begin{matrix} r \rightarrow X \\ \theta \rightarrow Y \end{matrix}$$

X, Y にして一般化

$$x = x(X, Y), y = y(X, Y)$$

$$x = x(X, Y), \quad y = y(X, Y)$$

$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt \Rightarrow \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \text{ を } \frac{dX}{dt}, \frac{dY}{dt} \text{ で表す}$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial X} \frac{dX}{dt} + \frac{\partial x}{\partial Y} \frac{dY}{dt} \quad \frac{dy}{dt} = \frac{\partial y}{\partial X} \frac{dX}{dt} + \frac{\partial y}{\partial Y} \frac{dY}{dt}$$

$$\left(\frac{dx}{dt} \right)^2 = \left(\frac{\partial x}{\partial X} \frac{dX}{dt} + \frac{\partial x}{\partial Y} \frac{dY}{dt} \right)^2 = \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dX}{dt} + 2 \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} \frac{dX}{dt} \frac{dY}{dt} + \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} \frac{dY}{dt} \frac{dY}{dt}$$

$$\left(\frac{dy}{dt} \right)^2 = \left(\frac{\partial y}{\partial X} \frac{dX}{dt} + \frac{\partial y}{\partial Y} \frac{dY}{dt} \right)^2 = \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dX}{dt} + 2 \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \frac{dX}{dt} \frac{dY}{dt} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \frac{dY}{dt} \frac{dY}{dt}$$

$$\begin{aligned} \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 &= \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} \frac{dX}{dt} \frac{dX}{dt} + 2 \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} \frac{dX}{dt} \frac{dY}{dt} + \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \\ &\quad + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} \frac{dX}{dt} \frac{dX}{dt} + 2 \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \frac{dX}{dt} \frac{dY}{dt} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \frac{dY}{dt} \frac{dY}{dt} \end{aligned}$$

$$= \left(\frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} \right) \frac{dX}{dt} \frac{dX}{dt} + 2 \left(\frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \right) \frac{dX}{dt} \frac{dY}{dt} + \left(\frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \right) \frac{dY}{dt} \frac{dY}{dt}$$

$$= \begin{pmatrix} \frac{dX}{dt} & \frac{dY}{dt} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix} \begin{pmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{pmatrix} \leftarrow \frac{d\mathbf{X}}{dt} = \begin{pmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{pmatrix}, \quad \frac{d\mathbf{X}^T}{dt} = \begin{pmatrix} \frac{dX}{dt} & \frac{dY}{dt} \end{pmatrix}$$

$$= \frac{d\mathbf{X}^T}{dt} \mathbf{g}(\mathbf{X}) \frac{d\mathbf{X}}{dt} \leftarrow \mathbf{g}(\mathbf{X}) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix} \rightarrow \begin{cases} \mathbf{g}^T(\mathbf{X}) = \mathbf{g}(\mathbf{X}) \\ \mathbf{g}(\mathbf{X})_{\alpha\beta} = \mathbf{g}(\mathbf{X})_{\beta\alpha} \end{cases}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}$$

$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} dt \Rightarrow \mathcal{L} = \sqrt{c^2 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}}$$

←ラグランジュ方程式: $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \left(\frac{d\mathbf{X}^\alpha}{dt}\right)} = \frac{\partial \mathcal{L}}{\partial \mathbf{X}^\alpha}$

$$\Rightarrow \frac{d}{dt} \frac{1}{2} \frac{2 \left(-\mathbf{g} \frac{d\mathbf{X}}{dt}\right)_\alpha}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}}} = \frac{1}{2} \frac{-\frac{d\mathbf{X}^T}{dt} \frac{\partial \mathbf{g}}{\partial \mathbf{X}^\alpha} \frac{d\mathbf{X}}{dt}}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}}} \Rightarrow \frac{d}{dt} \frac{2 \left(\mathbf{g} \frac{d\mathbf{X}}{dt}\right)_\alpha}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}}} = \frac{\frac{d\mathbf{X}^T}{dt} \frac{\partial \mathbf{g}}{\partial \mathbf{X}^\alpha} \frac{d\mathbf{X}}{dt}}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}}}$$

$$\Rightarrow 2 \frac{d}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}} dt} \left(\frac{\mathbf{g} \frac{d\mathbf{X}}{dt}}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}} dt} \right)_\alpha = \frac{\frac{d\mathbf{X}^T}{dt} \frac{\partial \mathbf{g}}{\partial \mathbf{X}^\alpha} \frac{d\mathbf{X}}{dt}}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}} dt} \frac{d\mathbf{X}}{\sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}} dt}$$

$$\Rightarrow 2 \frac{d}{cd\tau} \left(\mathbf{g} \frac{d\mathbf{X}}{cd\tau} \right)_\alpha = \frac{d\mathbf{X}^T}{cd\tau} \frac{\partial \mathbf{g}}{\partial \mathbf{X}^\alpha} \frac{d\mathbf{X}}{cd\tau} \Leftarrow cd\tau = \sqrt{c^2 - \frac{d\mathbf{X}^T}{dt} \mathbf{g} \frac{d\mathbf{X}}{dt}} dt$$

$$\frac{d}{cd\tau} \left(\mathbf{g} \frac{d\mathbf{X}}{cd\tau} \right)_\alpha = \frac{1}{2} \frac{d\mathbf{X}^T}{cd\tau} \frac{\partial \mathbf{g}}{\partial \mathbf{X}^\alpha} \frac{d\mathbf{X}}{cd\tau}$$

$$\frac{d^2 X^{\beta'}}{d^2 \tau} + \frac{1}{2} \left(\sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\alpha\beta}}{\partial X^\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} + \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\alpha\gamma}}{\partial X^\beta} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} - \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\beta\gamma}}{\partial X^\alpha} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} \right) = 0$$

$$\frac{d^2 X^{\beta'}}{d^2 \tau} + \frac{1}{2} \left(\sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\alpha\beta}}{\partial X^\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} + \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\alpha\gamma}}{\partial X^\beta} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} - \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{\partial \mathbf{g}_{\beta\gamma}}{\partial X^\alpha} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} \right) = 0$$

$$\Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} + \frac{1}{2} \left[\sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \left(\frac{\partial \mathbf{g}_{\alpha\beta}}{\partial X^\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} + \frac{\partial \mathbf{g}_{\alpha\gamma}}{\partial X^\beta} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} + \frac{\partial \mathbf{g}_{\beta\gamma}}{\partial X^\alpha} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} \right) \right] = 0$$

$$\Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \frac{1}{2} \left(\frac{\partial \mathbf{g}_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial \mathbf{g}_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial \mathbf{g}_{\beta\gamma}}{\partial X^\alpha} \right) \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

$$\Leftarrow \Gamma_{\alpha, \beta\gamma} = \frac{1}{2} \left(\frac{\partial \mathbf{g}_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial \mathbf{g}_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial \mathbf{g}_{\beta\gamma}}{\partial X^\alpha} \right)$$

$$\Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\alpha, \beta, \gamma=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

$$\Leftarrow \Gamma^{\beta'}_{\beta\gamma} = \sum_{\alpha=1}^2 (\mathbf{g}^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma}$$

$$\Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\beta, \gamma=1}^2 \Gamma^{\beta'}_{\beta\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

$$\frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\beta, \gamma=1}^2 \Gamma^{\beta'}_{\beta\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

測地線方程式

測地線方程式(相対性理論)

$$\frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\beta, \gamma=1}^2 \Gamma^{\beta'}_{\beta\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

$$\Gamma_{\alpha, \beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} \right), \Gamma^{\beta'}_{\beta\gamma} = \sum_{\alpha=1}^2 (g^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma}, g(X) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix}$$

【例】無重力=直交座標系

$$x = X, y = Y \Rightarrow g(X) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \Gamma_{\alpha, \beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} \right) = \text{すべて零} \Rightarrow \Gamma^{\beta'}_{\beta\gamma} = \sum_{\alpha=1}^2 (g^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma} = \text{すべて零}$$

$$\Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\beta, \gamma=1}^2 \overset{\text{すべて零}}{\Gamma^{\beta'}_{\beta\gamma}} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0 \Rightarrow \frac{d^2 X^{\beta'}}{d^2 \tau} = 0 \Rightarrow \frac{d^2 X}{d^2 \tau} = \frac{d^2 Y}{d^2 \tau} = 0$$

$$\Rightarrow X = \overset{\text{定数}}{\widetilde{A}} \tau + \overset{\text{定数}}{\widetilde{B}}, Y = \overset{\text{定数}}{\widetilde{C}} \tau + \overset{\text{定数}}{\widetilde{D}} \Rightarrow \frac{Y - \widetilde{D}}{X - \widetilde{B}} = \frac{\widetilde{C}}{\widetilde{A}}$$

$$\Rightarrow Y = \overset{=\frac{C}{A}}{\underset{\text{定数}}{\widetilde{a}}} X + \overset{=D - \frac{C}{A} \widetilde{B}}{\underset{\text{定数}}{\widetilde{b}}}$$

直線の方程式

測地線方程式(4次元相対性理論)

3次元相対性理論(空間2次元)

$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right]} dt$$

$$\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mathbf{X}^1 \\ \mathbf{X}^2 \end{pmatrix}$$

$$\alpha, \beta, \gamma = 1, 2 \Rightarrow \Gamma_{\alpha, \beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} \right) \quad \Leftarrow \alpha, \beta, \gamma = 1, 2, 3$$

$$\Gamma^{\beta'}_{\beta\gamma} = \sum_{\alpha=1}^2 (g^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma}$$

$$g(\mathbf{X}) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} \end{pmatrix}$$

4次元相対性理論(空間3次元)

$$S = \int_{t_i}^{t_f} \sqrt{c^2 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]} dt$$

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \mathbf{X}^1 \\ \mathbf{X}^2 \\ \mathbf{X}^3 \end{pmatrix}$$

$$\alpha, \beta, \gamma = 1, 2, 3 \Rightarrow \Gamma_{\alpha, \beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} \right) \quad \Leftarrow \alpha, \beta, \gamma = 1, 2, 3$$

$$\Gamma^{\beta'}_{\beta\gamma} = \sum_{\alpha=1}^3 (g^{-1})^{\beta'\alpha} \Gamma_{\alpha, \beta\gamma}$$

$$g(\mathbf{X}) = \begin{pmatrix} \frac{\partial x}{\partial X} \frac{\partial x}{\partial X} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial X} + \frac{\partial z}{\partial X} \frac{\partial z}{\partial X} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} + \frac{\partial z}{\partial X} \frac{\partial z}{\partial Y} & \frac{\partial x}{\partial X} \frac{\partial x}{\partial Z} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Z} + \frac{\partial z}{\partial X} \frac{\partial z}{\partial Z} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Y} + \frac{\partial z}{\partial X} \frac{\partial z}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Y} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Y} + \frac{\partial z}{\partial Y} \frac{\partial z}{\partial Y} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Z} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Z} + \frac{\partial z}{\partial Y} \frac{\partial z}{\partial Z} \\ \frac{\partial x}{\partial X} \frac{\partial x}{\partial Z} + \frac{\partial y}{\partial X} \frac{\partial y}{\partial Z} + \frac{\partial z}{\partial X} \frac{\partial z}{\partial Z} & \frac{\partial x}{\partial Y} \frac{\partial x}{\partial Z} + \frac{\partial y}{\partial Y} \frac{\partial y}{\partial Z} + \frac{\partial z}{\partial Y} \frac{\partial z}{\partial Z} & \frac{\partial x}{\partial Z} \frac{\partial x}{\partial Z} + \frac{\partial y}{\partial Z} \frac{\partial y}{\partial Z} + \frac{\partial z}{\partial Z} \frac{\partial z}{\partial Z} \end{pmatrix}$$

次元数によらない表記

$$\frac{d^2 X^{\beta'}}{d^2 \tau} + \sum_{\beta, \gamma=1}^2 \Gamma^{\beta'}_{\beta\gamma} \frac{dX^\gamma}{d\tau} \frac{dX^\beta}{d\tau} = 0$$