

第四章 インスタントン

ユークリッド計量では、

$$\int d^4x_M \mathcal{L}_M = i \int d^4x_E \mathcal{L}_E \begin{cases} \mathcal{L}_M = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \\ \mathcal{L}_E = \frac{1}{2} \text{Tr}(F_{\mu\nu E} F_E^{\mu\nu}) \end{cases}$$

Action、 S 、は、

$$W = \int [dA_\mu^a] \exp\left(i \int d^4x \mathcal{L}_M\right) = \int [dA_{\mu E}^a] \exp\left(i \int d^4x (i\mathcal{L})_E\right) = \int [dA_{\mu E}^a] \exp\left(-\int d^4x \mathcal{L}_E\right)$$

ここで、

$$\mathcal{L}_E = \frac{1}{2} \text{Tr}(F_{\mu\nu E} F_E^{\mu\nu}) = \frac{1}{2} \text{Tr}(2F_E^{4i} F_E^{4i} + F_E^{ij} F_E^{ij}) \geq 0$$

なので、

$$W = \int [dA_{\mu E}^a] \exp\left(-\int d^4x \mathcal{L}_E\right) = \int [dA_{\mu E}^a] \exp\left(-\int d^4x \frac{1}{2} \text{Tr}(F_{\mu\nu E} F_E^{\mu\nu})\right) < \infty$$

が保証される。

第一節 : Topological Charge

ここで、ユークリッド計量での $\tilde{F}_E^{\mu\nu}$ を定義する。

$$\tilde{F}_E^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu E} \left(= \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu E} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \tilde{F}_E^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_E^{\mu\nu} \right) (\varepsilon^{1234} = \varepsilon_{1234} = 1)$$

次の量は、ゲージ変換で不变である：

$$\nu = \frac{g^2}{16\pi^2} \int d^4x_E \text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E}) = \text{ゲージ不变量}$$

実は、 ν 自身は任意の場の変換 ($\delta A_E^{a\mu}$) に際しても不变になる。すなわち、

$$\frac{\delta \nu}{\delta A_E^\mu} = 0$$

である。

【理由】 $F_E^{\mu\nu} = \partial^\mu A_E^\nu - \partial^\nu A_E^\mu - ig [A_E^\mu, A_E^\nu]$

$$F_E^{\mu\nu} = \Lambda^a \mathbf{F}_E^{a\mu\nu}, A_E^\mu = \Lambda^a \mathbf{A}_E^{a\mu} \Rightarrow [\Lambda^a, \Lambda^b] = if^{abc} \Lambda^c, \text{Tr}(\Lambda^a \Lambda^b) = \frac{1}{2} \delta^{ab} \quad (*) \quad \Lambda^a = \frac{\tau^a}{2} \text{ for SU(2)}$$

$$\begin{aligned} \Lambda^a \mathbf{F}_E^{a\mu\nu} &= \partial^\mu \Lambda^a \mathbf{A}_E^{a\nu} - \partial^\nu \Lambda^a \mathbf{A}_E^{a\mu} - ig [\Lambda^b \mathbf{A}_E^{b\mu}, \Lambda^c \mathbf{A}_E^{c\nu}] = \partial^\mu \Lambda^a A_E^{a\nu} - \partial^\nu \Lambda^a A_E^{a\mu} - ig f^{bca} \Lambda^a \mathbf{A}_E^{b\mu} \mathbf{A}_E^{c\nu} \\ &\Rightarrow \mathbf{F}_E^{a\mu\nu} = \partial^\mu \mathbf{A}_E^{a\nu} - \partial^\nu \mathbf{A}_E^{a\mu} + g f^{abc} \mathbf{A}_E^{b\mu} \mathbf{A}_E^{c\nu} \end{aligned}$$

$$\frac{\delta A_E^{bv}(x)}{\delta A_E^{a\mu}(y)} = \delta_a^b \delta_\mu^\nu \delta(x-y)$$

$$\begin{aligned}
\frac{\delta F_E^{b\nu\rho}(x)}{\delta A_E^{a\mu}(y)} &= \frac{\delta}{\delta A_E^{a\mu}(y)} \left(\partial_x^\nu A_E^{b\rho}(x) - \partial_x^\rho A_E^{b\nu}(x) + g f^{bcd} A_E^{cv}(x) A_E^{d\rho}(x) \right) \\
&= \partial_y^\nu \delta_a^b \delta_\mu^\rho \delta(x-y) - \partial_y^\rho \delta_a^b \delta_\mu^\nu \delta(x-y) + g f^{bcd} \delta_a^c \delta_\mu^\nu \delta(x-y) A_E^{d\rho}(x) + g f^{bcd} A_E^{cv}(x) \delta_a^d \delta_\mu^\rho \delta(x-y) \\
&= \left[\partial_x^\nu \delta_a^b \delta_\mu^\rho + g f^{bca} A_E^{cv}(x) \delta_\mu^\rho \right] \delta(x-y) - \left[\partial_x^\rho \delta_a^b \delta_\mu^\nu - g f^{bad} \delta_\mu^\nu A_E^{d\rho}(x) \right] \delta(x-y) \\
&= \left[\partial_x^\nu \delta_a^b + g f^{bca} A_E^{cv}(x) \right] \delta_\mu^\rho \delta(x-y) - \left[\partial_x^\rho \delta_a^b - g f^{bad} A_E^{d\rho}(x) \right] \delta_\mu^\nu \delta(x-y) \\
&\Leftarrow \left(\Lambda_{ADJ}^c \right)^{ba} = -if^{cba} = if^{bca}, \left(\Lambda_{ADJ}^d \right)^{ba} = -if^{dba} \text{ for the adjoint representation} \\
&= \left[\partial_x^\nu \delta_a^b + g \left(-i \Lambda_{ADJ}^c \right)^{ba} A_E^{cv}(x) \right] \delta_\mu^\rho \delta(x-y) - \left[\partial_x^\rho \delta_a^b - g \left(i \Lambda_{ADJ}^d \right)^{ba} A_E^{d\rho}(x) \right] \delta_\mu^\nu \delta(x-y) \\
&= \left[\partial_x^\nu \delta_a^b - g \left(i \Lambda_{ADJ}^c A_E^{cv}(x) \right)^{ba} \right] \delta_\mu^\rho \delta(x-y) - \left[\partial_x^\rho \delta_a^b - g \left(i \Lambda_{ADJ}^d A_E^{d\rho}(x) \right)^{ba} \right] \delta_\mu^\nu \delta(x-y) \\
&\Leftarrow A_E^\mu = \Lambda_{ADJ}^a A_E^{a\mu} \\
&= \left[\partial_x^\nu \delta_a^b - ig A_E^\nu(x)^{ba} \right] \delta_\mu^\rho \delta(x-y) - \left[\partial_x^\rho \delta_a^b - ig A_E^\rho(x)^{ba} \right] \delta_\mu^\nu \delta(x-y) \\
&\Leftarrow D^\mu = \partial^\mu - ig A_E^\mu \\
&= D_E^{\nu ba} \delta_\mu^\rho \delta(x-y) - D_E^{\rho ba} \delta_\mu^\nu \delta(x-y)
\end{aligned}$$

従って、

$$\begin{aligned}
\frac{\delta v}{\delta A_E^{a\mu}(y)} &= \frac{\delta}{\delta A_E^{a\mu}(y)} \left(\frac{g^2}{16\pi^2} \int d^4 x_E \text{Tr} \left(F_E^{\nu\rho} \tilde{F}_{\nu\rho E} \right) \right) = \frac{1}{2} \epsilon_{\nu\rho\nu'\rho'} \frac{g^2}{16\pi^2} \int d^4 x_E \frac{\delta}{\delta A_E^{a\mu}(y)} \text{Tr} \left(F_E^{\nu\rho} F_E^{\nu'\rho'} \right) \\
&= \epsilon_{\nu\rho\nu'\rho'} \text{Tr} \left(\Lambda^b \Lambda^{b'} \right) \frac{g^2}{32\pi^2} \int d^4 x_E \frac{\delta}{\delta A_E^{a\mu}(y)} F_E^{b\nu\rho} F_E^{b'\nu'\rho'} \\
&\Leftarrow \frac{\delta F_E^{b\nu\rho}(x)}{\delta A_E^{a\mu}(y)} = D_E^{\nu ba} \delta_\mu^\rho \delta(x-y) - D_E^{\rho ba} \delta_\mu^\nu \delta(x-y) \\
&\Leftarrow \text{Tr} \left(\Lambda^a \Lambda^b \right) = \frac{1}{2} \delta^{ab} \\
&= \epsilon_{\nu\rho\nu'\rho'} \frac{g^2}{64\pi^2} \left(\int d^4 x_E \left(D_E^{\nu ba} \delta_\mu^\rho \delta(x-y) - D_E^{\rho ba} \delta_\mu^\nu \delta(x-y) \right) F_E^{b'\nu'\rho'} \right. \\
&\quad \left. + \int d^4 x_E F_E^{b\nu\rho} \left(D_E^{\nu ba} \delta_\mu^{\rho'} \delta(x-y) - D_E^{\rho' ba} \delta_\mu^{\nu'} \delta(x-y) \right) \right) \\
&= \frac{g^2}{64\pi^2} \left(\int d^4 x_E \left(\epsilon_{\nu\mu\nu'\rho'} D_E^{\nu ba} \delta(x-y) - \epsilon_{\mu\rho\nu'\rho'} D_E^{\rho ba} \delta(x-y) \right) F_E^{b'\nu'\rho'} \right. \\
&\quad \left. + \int d^4 x_E F_E^{b\nu\rho} \left(\epsilon_{\nu\rho\nu'\mu} D_E^{\nu ba} \delta(x-y) - \epsilon_{\nu\rho\mu\nu'} D_E^{\rho' ba} \delta(x-y) \right) \right) \\
&= \frac{4g^2}{64\pi^2} \int d^4 x_E \left(\epsilon_{\nu\mu\nu'\rho'} D_E^{\nu ba} \delta(x-y) \right) F_E^{b'\nu'\rho'} \\
&= \frac{g^2}{16\pi^2} \epsilon_{\nu\mu\nu'\rho'} \int d^4 x_E D_E^{\nu ba} \delta(x-y) \cdot F_E^{b'\nu'\rho'} \stackrel{D^\mu = \partial^\mu - ig A_E^\mu}{=} \frac{g^2}{16\pi^2} \epsilon_{\nu\mu\nu'\rho'} \int d^4 x_E \left(\partial^\nu - ig A_E^\nu \right)^{ba} \delta(x-y) \cdot D_E^{b'\nu'\rho'} \\
&= \frac{g^2}{16\pi^2} \epsilon_{\nu\mu\nu'\rho'} \left[\int d^4 x_E \delta(x-y) \left(-\partial^\nu - ig A_E^\nu \right)^{ba} F_E^{b'\nu'\rho'} + \int d^4 x_E \partial^\nu \left(\delta(x-y) \delta^{ba} F_E^{b'\nu'\rho'} \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{g^2}{16\pi^2} \varepsilon_{\nu\mu\nu'\rho'} \left[\int d^4x_E \delta(x-y) \cdot (-\partial^\nu - ig A_E^\nu)^{ba} F_E^{bv'\rho'} + \int d\sigma_E^\nu \delta(x-y) \cdot F_E^{av'\rho'}(y) \Big|_{r=\infty} \right] \\
&\Leftarrow \nu = 0 : \int d\sigma_E^\nu \delta(x-y) \cdot F_E^{av'\rho'}(y) \Big|_{r=\infty} = \int dx \delta(x-y) \cdot F_E^{av'\rho'}(y) \Big|_{r=\infty} = \delta(x^0 - y^0) F_E^{av'\rho'}(y) \Big|_{r=\infty} \\
&= \begin{cases} 0 & \text{for } x^0 \neq y^0 \\ F_E^{av'\rho'}(y) \Big|_{r=\infty} & = 0 \text{ for } x^0 = y^0 \end{cases} \\
&\text{等} \\
&= \frac{g^2}{16\pi^2} \varepsilon_{\nu\mu\nu'\rho'} (-\partial^\nu \square [ig A_E^\nu])^{ba} F_E^{bv'\rho'}(y) \stackrel{a \leftrightarrow b}{=} \frac{g^2}{16\pi^2} \varepsilon_{\nu\mu\nu'\rho'} (-\partial^\nu \square [ig A_E^\nu])^{ab} F_E^{bv'\rho'}(y) \\
&= \frac{g^2}{16\pi^2} (-D^{\nu ab}) \frac{1}{2} \varepsilon_{\nu\mu\nu'\rho'} F_E^{bv'\rho'}(y) = \frac{g^2}{16\pi^2} (-D^{\nu ab}) \tilde{F}_{\nu\mu E}^b(y) = -\frac{g^2}{16\pi^2} D^{\nu ab} \tilde{F}_{\nu\mu E}^b(y) \\
&= -\frac{g^2}{16\pi^2} (D^\nu \tilde{F}_{\nu\mu E}(y))^a = \frac{g^2}{16\pi^2} (D^\nu \tilde{F}_{\mu\nu E}(y))^a
\end{aligned}$$

従つて、

$$\boxed{\frac{\delta v}{\delta A_E^{a\mu}(y)} = \frac{g^2}{16\pi^2} (D^\nu \tilde{F}_{\mu\nu E}(y))^a}$$

$$\begin{aligned}
(D^\nu \tilde{F}_{\mu\nu E}(y))^a &\stackrel{D^\mu = \partial^\mu - ig A_E^\mu}{=} (\partial^\nu - ig A_E^\nu)^{ab} \tilde{F}_{\mu\nu E}^b(y) = \left(\partial^\nu \delta^{ab} - ig (\Lambda_{ADJ}^c)^{ab} A_E^{cv} \right) \tilde{F}_{\mu\nu E}^b(y) \\
&\Leftarrow (\Lambda_{ADJ}^c)^{ab} = -if^{cab} \\
&= (\partial^\nu \delta^{ab} - ig (-if^{cab}) A_E^{cv}) \tilde{F}_{\mu\nu E}^b(y) \\
&\Leftarrow [\Lambda^a, \Lambda^b] = if^{abc} \Lambda_{ADJ}^c \\
(D^\nu \tilde{F}_{\mu\nu E}(y))^a \Lambda^a &= (\partial^\nu \delta^{ab} - ig (-if^{cab}) A_E^{cv}) \Lambda^a \tilde{F}_{\mu\nu E}^b(y) = (\partial^\nu \Lambda^b - gf^{cab} \Lambda^a A_E^{cv}) \tilde{F}_{\mu\nu E}^b(y) \\
&= (\partial^\nu \Lambda^b - gf^{bca} \Lambda^a A_E^{cv}) \tilde{F}_{\mu\nu E}^b(y) = (\partial^\nu \Lambda^b - g(-i)[\Lambda^b, \Lambda^c] A_E^{cv}) \tilde{F}_{\mu\nu E}^b(y) \\
&\stackrel{b \leftrightarrow c}{=} (\partial^\nu \Lambda^b - ig[\Lambda^c, \Lambda^b] A_E^{cv}) \tilde{F}_{\mu\nu E}^b(y) = \partial^\nu \tilde{F}_{\mu\nu E}(y) - ig[A_E^\nu, \tilde{F}_{\mu\nu E}(y)]
\end{aligned}$$

これより、

$$\begin{aligned}
(D^\nu \tilde{F}_{\mu\nu E}(y))^a \Lambda^a &= \partial^\nu \tilde{F}_{\mu\nu E} - ig[A_E^\nu, \tilde{F}_{\mu\nu E}] \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} (\partial^\nu F_E^{\rho\sigma} - ig[A_E^\nu, F_E^{\rho\sigma}]) \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left(\partial^\nu \left(\cancel{\partial^\rho A_E^\sigma} - \cancel{\partial^\sigma A_E^\rho} - ig[A_E^\rho, A_E^\sigma] \right) - ig[A_E^\nu, \partial^\rho A_E^\sigma - \partial^\sigma A_E^\rho - ig[A_E^\rho, A_E^\sigma]] \right) \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left(-ig \partial^\nu [A_E^\rho, A_E^\sigma] - ig[A_E^\nu, \partial^\rho A_E^\sigma - \partial^\sigma A_E^\rho - ig[A_E^\rho, A_E^\sigma]] \right) \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left((-ig) (\partial^\nu [A_E^\rho, A_E^\sigma] + [A_E^\nu, \partial^\rho A_E^\sigma - \partial^\sigma A_E^\rho]) - g^2 [A_E^\nu, [A_E^\rho, A_E^\sigma]] \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left((-ig) \left([\partial^\nu A_E^\rho, A_E^\sigma] + [A_E^\rho, \partial^\nu A_E^\sigma] + [A_E^\nu, \partial^\rho A_E^\sigma] - [A_E^\nu, \partial^\sigma A_E^\rho] \right) - g^2 [A_E^\nu, [A_E^\rho, A_E^\sigma]] \right) \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left((-ig) \left(\cancel{[\partial^\nu A_E^\rho, A_E^\sigma]} + \cancel{[A_E^\rho, \partial^\nu A_E^\sigma]} - \cancel{[A_E^\nu, \partial^\rho A_E^\sigma]} + \cancel{[A_E^\sigma, \partial^\nu A_E^\rho]} \right) - g^2 [A_E^\nu, [A_E^\rho, A_E^\sigma]] \right) \\
&= \frac{1}{2} \varepsilon_{\nu\mu\rho\sigma} \left(-g^2 [A_E^\nu, [A_E^\rho, A_E^\sigma]] \right) \Leftarrow \nu[\rho, \sigma] \equiv [A_E^\nu, [A_E^\rho, A_E^\sigma]] : \\
&\Leftarrow \nu[\rho, \sigma] \xrightarrow{\varepsilon_{\nu\mu\rho\sigma}} \overline{\nu[\rho, \sigma] - \nu[\sigma, \rho]} \xrightarrow{\varepsilon_{\nu\mu\rho\sigma}} \overline{\nu[\rho, \sigma] - \rho[\nu, \sigma] - \sigma[\rho, \nu] + \nu[\sigma, \rho] + \rho[\sigma, \nu]} \\
&= \nu[\rho, \sigma] + \sigma[\nu, \rho] + \rho[\sigma, \nu] - \left(\rho[\nu, \sigma] + \sigma[\rho, \nu] + \nu[\sigma, \rho] \right) \\
&= 0
\end{aligned}$$

つまり、

$$\boxed{\mathbf{D}^\nu \tilde{\mathbf{F}}_{\mu\nu E} = 0}$$

以上から、

$$\boxed{\frac{\delta \nu}{\delta A_E^{a\mu}(y)} = \frac{g^2}{16\pi^2} (\mathbf{D}^\nu \tilde{\mathbf{F}}_{\mu\nu E}(y))^a = 0}$$

つまり、 ν 自身はどのような局所変換に際しても（恒等的に）不変になる。この性質を

- 位相不变性 (Topological invariant)

という。

局所変換以外の寄与は、大局的変換になるが、そのような寄与は表面積分から現れる。表面積分の効果を探るために $\text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E}) \rightarrow \partial^\mu X_\mu$ ような X_μ を調べればよい：

$$\nu = \frac{g^2}{16\pi^2} \int d^4 x_E \text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E}) \rightarrow \int d^4 x_E \partial^\mu X_\mu = \int d\sigma^\mu X_\mu \text{ at infinity}$$

である。そこで

$$F_E^{\mu\nu} = \partial^\mu A_E^\nu - \partial^\nu A_E^\mu - ig [A_E^\mu, A_E^\nu] = (\partial^\mu - ig A_E^\mu) A_E^\nu - (\partial^\nu - ig A_E^\nu) A_E^\mu$$

を用いて、

$$\begin{aligned}
\text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E}) &= \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(F_E^{\mu\nu} F_E^{\rho\sigma}) = \frac{1}{2} 4\varepsilon_{\mu\nu\rho\sigma} \text{Tr}((\partial^\mu - ig A_E^\mu) A_E^\nu \cdot (\partial^\rho - ig A_E^\rho) A_E^\sigma) \\
&= 2\varepsilon_{\mu\nu\rho\sigma} \text{Tr}(\partial^\mu A_E^\nu \cdot \partial^\rho A_E^\sigma - 2ig \partial^\mu A_E^\nu \cdot A_E^\rho A_E^\sigma - g^2 A_E^\mu A_E^\nu A_E^\rho A_E^\sigma) \\
&= 2\varepsilon_{\mu\nu\rho\sigma} \text{Tr}(\partial^\mu A_E^\nu \cdot \partial^\rho A_E^\sigma - 2ig \partial^\mu A_E^\nu \cdot A_E^\rho A_E^\sigma) - 2\varepsilon_{\mu\nu\rho\sigma} g^2 \underbrace{\text{Tr}(A_E^\mu A_E^\nu A_E^\rho A_E^\sigma)}_{\substack{\text{symmetric in } \mu \rightarrow \nu \rightarrow \rho \rightarrow \sigma \\ \varepsilon_{\mu\nu\rho\sigma} \text{: antisymmetric in } \mu \rightarrow \nu \rightarrow \rho \rightarrow \sigma}} \\
&= 2\varepsilon_{\mu\nu\rho\sigma} \text{Tr}(\partial^\mu A_E^\nu \cdot \partial^\rho A_E^\sigma - 2ig \partial^\mu A_E^\nu \cdot A_E^\rho A_E^\sigma)
\end{aligned}$$

$$\begin{aligned}
&\Leftarrow \varepsilon_{\mu\nu\rho\sigma} \partial^\mu A_E^\nu \cdot \partial^\rho A_E^\sigma = \varepsilon_{\mu\nu\rho\sigma} \left[\partial^\mu \left(A_E^\nu \partial^\rho A_E^\sigma \right) - A_E^\nu \left(\begin{array}{cc} \cancel{\partial^\mu \partial^\rho} & A_E^\sigma \\ \text{symmetric in } \mu, \rho & \\ \varepsilon_{\mu\nu\rho\sigma} : \text{antisymmetric} & \end{array} \right) \right] = \varepsilon_{\mu\nu\rho\sigma} \partial^\mu \left(A_E^\nu \partial^\rho A_E^\sigma \right) \\
&\Leftarrow \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\boxed{\partial^\mu A_E^\nu \cdot A_E^\rho A_E^\sigma} \right) = \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[\partial^\mu \left(A_E^\nu A_E^\rho A_E^\sigma \right) - A_E^\nu \partial^\mu A_E^\rho \cdot A_E^\sigma - A_E^\nu A_E^\rho \partial^\mu A_E^\sigma \right] \\
&= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[\partial^\mu \left(A_E^\nu A_E^\rho A_E^\sigma \right) + A_E^\rho \partial^\mu A_E^\nu \cdot A_E^\sigma + A_E^\nu A_E^\sigma \partial^\mu A_E^\rho \right] \\
&= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[\partial^\mu \left(A_E^\nu A_E^\rho A_E^\sigma \right) - A_E^\sigma \partial^\mu A_E^\nu \cdot A_E^\rho - A_E^\rho A_E^\sigma \partial^\mu A_E^\nu \right] \\
&= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[\partial^\mu \left(A_E^\nu A_E^\rho A_E^\sigma \right) - 2 \boxed{\partial^\mu A_E^\nu \cdot A_E^\rho A_E^\sigma} \right] \\
&= 2 \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\partial^\mu \left(A_E^\nu \partial^\rho A_E^\sigma \right) - \frac{2ig}{3} \partial^\mu \left(A_E^\nu A_E^\rho A_E^\sigma \right) \right) = 4 \varepsilon_{\mu\nu\rho\sigma} \partial^\mu \text{Tr} \left(\frac{1}{2} A_E^\nu \partial^\rho A_E^\sigma - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right)
\end{aligned}$$

ここで、

$$\boxed{X_\mu = \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{2} A_E^\nu \partial^\rho A_E^\sigma - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right)}$$

とすると、

$$\boxed{\text{Tr} \left(F_E^{\mu\nu} \tilde{F}_{\mu\nu E} \right) = 4 \partial^\mu X_\mu}$$

従って、

$$\nu = \frac{g^2}{16\pi^2} \int d^4x_E \text{Tr} \left(F_E^{\mu\nu} \tilde{F}_{\mu\nu E} \right) = \frac{g^2}{4\pi^2} \int d^4x_E \partial^\mu X_\mu = \frac{g^2}{4\pi^2} \int d\sigma^\mu X_\mu \rightarrow 0 \text{ if } X_\mu \rightarrow 0 \text{ at infinity}$$

このような、単純な議論が成り立たないときには、

$$\int d\sigma^\mu X_\mu \rightarrow \text{cst} \text{ if } X_\mu \rightarrow \frac{1}{r^3} \text{ at infinity}$$

の条件が出る。そこで、この条件が成立するときを調べることになる。つまり

- $\bullet \quad X_\mu = \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{2} A_E^\nu \partial^\rho A_E^\sigma - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right)$ なので大雑把に、 $X_\mu \rightarrow \frac{1}{r^3}$ as $A_E^\mu \rightarrow \frac{1}{r}$ at infinity

と期待される。ちなみに、 X_μ のゲージ変換性を調べてみる。ゲージ変換は

$$F'_{\mu\nu} = UF_{\mu\nu}U^\dagger, \quad A'_\mu(x) = UA_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

である。また、

$$\begin{aligned}
X_\mu &= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{2} A_E^\nu \partial^\rho A_E^\sigma - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right) \\
&= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} A_E^\nu \left(\partial^\rho A_E^\sigma - \partial^\sigma A_E^\rho - ig [A_E^\rho, A_E^\sigma] \right) + \frac{1}{4} ig A_E^\nu [A_E^\rho, A_E^\sigma] - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right) \\
&= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} A_E^\nu F_E^{\rho\sigma} + \frac{1}{2} ig A_E^\nu A_E^\rho A_E^\sigma - \frac{1}{3} ig A_E^\nu A_E^\rho A_E^\sigma \right) = \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} A_E^\nu F_E^{\rho\sigma} + \frac{1}{6} ig A_E^\nu A_E^\rho A_E^\sigma \right)
\end{aligned}$$

なので、ゲージ変換は

$$\begin{aligned} X'_\mu &= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} A_E^\nu F_E^{\rho\sigma} + \frac{1}{6} ig A_E'^\nu A_E'^\rho A_E'^\sigma \right) \\ &= \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} \left(UA_E^\nu U^\dagger + \frac{i}{g} U \partial^\nu U^\dagger \right) U F_E^{\rho\sigma} U^\dagger \right. \\ &\quad \left. + \frac{1}{6} ig \left(UA_E^\nu U^\dagger + \frac{i}{g} U \partial^\nu U^\dagger \right) \left(UA_E^\rho U^\dagger + \frac{i}{g} U \partial^\rho U^\dagger \right) \left(UA_E^\sigma U^\dagger + \frac{i}{g} U \partial^\sigma U^\dagger \right) \right) \end{aligned}$$

第一項 :

$$\varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\frac{1}{4} U \left(A_E^\nu F_E^{\rho\sigma} + \frac{i}{g} \partial^\nu U^\dagger \cdot UF_E^{\rho\sigma} \right) U^\dagger \right) = \varepsilon_{\mu\nu\rho\sigma} \frac{1}{4} \text{Tr} \left(A_E^\nu F_E^{\rho\sigma} + \frac{i}{g} \partial^\nu U^\dagger \cdot UF_E^{\rho\sigma} \right)$$

第二項 :

$$\begin{aligned} &\varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \text{Tr} \left(UA_E^\nu U^\dagger + \frac{i}{g} U \partial^\nu U^\dagger \right) \left(UA_E^\rho U^\dagger + \frac{i}{g} U \partial^\rho U^\dagger \right) \left(UA_E^\sigma U^\dagger + \frac{i}{g} U \partial^\sigma U^\dagger \right) \\ &= \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \text{Tr} \left(UA_E^\nu U^\dagger UA_E^\rho U^\dagger UA_E^\sigma U^\dagger \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \frac{i}{g} \text{Tr} \left(U \partial^\nu U^\dagger \cdot UA_E^\rho U^\dagger UA_E^\sigma U^\dagger + UA_E^\nu U^\dagger U \partial^\rho U^\dagger \cdot UA_E^\sigma U^\dagger + UA_E^\nu U^\dagger UA_E^\rho U^\dagger U \partial^\sigma U^\dagger \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \left(\frac{i}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot UA_E^\sigma U^\dagger + UA_E^\nu U^\dagger U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger + U \partial^\nu U^\dagger \cdot UA_E^\rho U^\dagger U \partial^\sigma U^\dagger \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \left(\frac{i}{g} \right)^3 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\ &= \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \text{Tr} \left(A_E^\nu A_E^\rho A_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot UA_E^\rho A_E^\sigma + \partial^\rho U^\dagger \cdot UA_E^\sigma A_E^\nu + \partial^\sigma U^\dagger \cdot UA_E^\nu A_E^\rho \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \left(\frac{i}{g} \right)^2 \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot UA_E^\sigma + \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \cdot UA_E^\nu + \partial^\sigma U^\dagger \cdot U \partial^\nu U^\dagger \cdot UA_E^\rho \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \left(\frac{i}{g} \right)^3 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\ &= \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \text{Tr} \left(A_E^\nu A_E^\rho A_E^\sigma \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \text{Tr} \left(\partial^\nu U^\dagger \cdot UA_E^\rho A_E^\sigma \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot UA_E^\sigma \right) \\ &\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \end{aligned}$$

第一項+第二項 :

$$\begin{aligned} &\varepsilon_{\mu\nu\rho\sigma} \frac{1}{4} \text{Tr} \left(A_E^\nu F_E^{\rho\sigma} + \frac{i}{g} \partial^\nu U^\dagger \cdot UF_E^{\rho\sigma} \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} ig \text{Tr} \left(A_E^\nu A_E^\rho A_E^\sigma \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \text{Tr} \left(\partial^\nu U^\dagger \cdot UA_E^\rho A_E^\sigma \right) \\ &- \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot UA_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \end{aligned}$$

$$\begin{aligned}
&= X_\mu + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{4} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \left(\partial^\rho A_E^\sigma - \partial^\rho A_E^\sigma - ig [A_E^\rho, A_E^\sigma] \right) \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \text{Tr} \left(\partial^\nu U^\dagger \cdot U A_E^\rho A_E^\sigma \right) \\
&\quad - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U A_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\
&= X_\mu + \varepsilon_{\mu\nu\rho\sigma} \frac{2}{4} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \left(\partial^\rho A_E^\sigma - ig \cancel{A_E^\rho} \cancel{A_E^\sigma} \right) \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \cancel{A_E^\rho} \cancel{A_E^\sigma} \right) \\
&\quad - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U A_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\
&= X_\mu + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\partial^\nu U^\dagger \cdot U \partial^\rho A_E^\sigma \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \partial^\nu \text{Tr} \left(U^\dagger \cdot U \partial^\rho U^\dagger \cdot U A_E^\sigma \right) \\
&\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\cancel{U^\dagger} \cdot U \partial^\nu \partial^\rho U^\dagger \cdot \cancel{U A_E^\sigma} \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\nu A_E^\sigma \right) \\
&\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\
&= X_\mu + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\cancel{\partial^\nu U^\dagger} \cdot \cancel{U \partial^\rho A_E^\sigma} \right) - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \partial^\nu \text{Tr} \left(\partial^\rho U^\dagger \cdot U A_E^\sigma \right) \\
&\quad + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \text{Tr} \left(\cancel{\partial^\rho U^\dagger} \cdot \cancel{U \partial^\nu A_E^\sigma} \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\
&= X_\mu - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \partial^\nu \text{Tr} \left(\partial^\rho U^\dagger \cdot U A_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right)
\end{aligned}$$

以上から、

$$\boxed{X'_\mu = X_\mu - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \partial^\nu \text{Tr} \left(\partial^\rho U^\dagger \cdot U A_E^\sigma \right) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right)}$$

を得る。

物理は作用積分 (S_E とする)

$$\begin{aligned}
S_E &= \int d^4x \frac{1}{2} \text{Tr} \left(F_{\mu\nu E} F_E^{\mu\nu} \right) \\
W &= \int [dA_{\mu E}^a] \exp(-S_E) = \int [dA_{\mu E}^a] \exp \left(- \int d^4x \mathcal{L}_E \right) = \int [dA_{\mu E}^a] \exp \left(- \int d^4x \frac{1}{2} \text{Tr} \left(F_{\mu\nu E} F_E^{\mu\nu} \right) \right)
\end{aligned}$$

で表されるのだが、もちろんこれは任意の場の変換 ($\delta A_E^{a\mu}$) に際して不变でない：

$$\begin{aligned}
\frac{\delta S_E}{\delta A_E^{a\mu}(y)} &= \frac{\delta}{\delta A_E^{a\mu}(y)} \left(\frac{1}{2} \int d^4x \text{Tr} \left(F_E^{\nu\rho} F_{\nu\rho E} \right) \right) = \text{Tr} \left(\Lambda^b \Lambda^{b'} \right) \frac{1}{2} \int d^4x \frac{\delta}{\delta A_E^{a\mu}(y)} \mathbf{F}_E^{b\rho} \mathbf{F}_{\nu\rho E}^{b'} \\
&\Leftarrow \frac{\delta \mathbf{F}_E^{b\rho}(x)}{\delta A_E^{a\mu}(y)} = \mathbf{D}_E^{vb} \delta_\mu^\rho \delta(x-y) - \mathbf{D}_E^{\rho ba} \delta_\mu^\nu \delta(x-y) \\
&\Leftarrow \text{Tr} \left(\Lambda^a \Lambda^b \right) = \frac{1}{2} \delta^{ab}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1}{2} 2 \int d^4 x_E \left(\mathbf{D}_E^{\nu ba} \delta_\mu^\rho \delta(x-y) - \mathbf{D}_E^{\rho ba} \delta_\mu^\nu \delta(x-y) \right) \mathbf{F}_{\nu\rho E}^b \\
&= \frac{1}{2} \frac{1}{2} 2 \cdot 2 \int d^4 x_E \mathbf{D}_E^{\nu ba} \delta(x-y) \cdot \mathbf{F}_{\nu\rho E}^b \\
&= \int d^4 x_E \mathbf{D}_E^{\nu ba} \delta(x-y) \cdot \mathbf{F}_{\nu\rho E}^b \stackrel{D^\mu = \partial^\mu - ig A_E^\mu}{=} \int d^4 x_E \left(\partial^\nu - ig A_E^\nu \right)^{ba} \delta(x-y) \cdot \mathbf{F}_{\nu\rho E}^b \\
&\stackrel{\text{部分積分}}{=} \int d^4 x_E \delta(x-y) \left(-\bar{\partial}^\nu - ig A_E^\nu \right)^{ba} \mathbf{F}_{\nu\rho E}^b = \left(-\bar{\partial}^\nu - ig A_E^\nu \right)^{ba} \mathbf{F}_{\nu\rho E}^b \\
&\stackrel{A_E^{vba} = -A_E^{vab}}{=} \left(-\bar{\partial}^\nu + ig A_E^\nu \right)^{ab} \mathbf{F}_{\nu\rho E}^b = -\mathbf{D}_E^{vab} \mathbf{F}_{\nu\rho E}^b = -\left(\mathbf{D}_E^\nu \mathbf{F}_{\nu\rho E} \right)^a
\end{aligned}$$

従って、

$$\boxed{\frac{\delta S_E}{\delta A_E^{a\mu}(y)} = -\left(\mathbf{D}_E^\nu \mathbf{F}_{\nu\rho E} \right)^a}$$

になり、これはゼロでないので、「位相不変量」ではない。

第二節 : Pure Gauge 解

さて、

$$\left(\mathbf{F}_{\mu\nu E}^a \pm \tilde{\mathbf{F}}_{\mu\nu E}^a \right)^2 \geq 0 \Leftrightarrow \mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} + \tilde{\mathbf{F}}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} \geq \mp 2 \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu}$$

上り、

$$\begin{aligned}
\tilde{\mathbf{F}}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} &= \left(\frac{1}{2} \right)^2 \epsilon_{\mu\nu\mu'\nu'} \epsilon^{\mu\nu\mu''\nu''} \mathbf{F}_{\mu''\nu'' E}^a \mathbf{F}_E^{a\mu'\nu'} \\
&\Leftarrow \epsilon_{\mu\nu\mu'\nu'} \epsilon^{\mu\nu\mu''\nu''} = 2! \left(\delta_{\mu'}^{\mu''} \delta_{\nu'}^{\nu''} - \delta_{\mu'}^{\nu''} \delta_{\nu'}^{\mu''} \right) \\
&= \frac{2!}{4} \left(\delta_{\mu'}^{\mu''} \delta_{\nu'}^{\nu''} - \delta_{\mu'}^{\nu''} \delta_{\nu'}^{\mu''} \right) \mathbf{F}_{\mu''\nu'' E}^a \mathbf{F}_E^{a\mu'\nu'} = \frac{1}{2} \left(\mathbf{F}_{\mu'\nu' E}^a \mathbf{F}_E^{a\mu'\nu'} - \mathbf{F}_{\mu'\nu' E}^a \mathbf{F}_E^{a\nu'\mu'} \right) = \mathbf{F}_{\mu'\nu' E}^a \mathbf{F}_E^{a\mu'\nu'}
\end{aligned}$$

なので、

$$\begin{aligned}
\mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} + \tilde{\mathbf{F}}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} &= 2 \mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} \geq \mp 2 \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} \Rightarrow \mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} \geq \mp \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} \\
\Rightarrow \mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} &\geq \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu}, -\mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} \leq \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu}
\end{aligned}$$

従って、

$$-\mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu} \leq \mathbf{F}_{\mu\nu E}^a \tilde{\mathbf{F}}_E^{a\mu\nu} \leq \mathbf{F}_{\mu\nu E}^a \mathbf{F}_E^{a\mu\nu}$$

が成り立つ。故に、

$$S_E = \int d^4 x \frac{1}{2} \text{Tr} \left(F_{\mu\nu E} F_E^{\mu\nu} \right) \geq \int d^4 x \frac{1}{2} \left| \text{Tr} \left(F_{\mu\nu E} \tilde{F}_E^{\mu\nu} \right) \right| = \frac{8\pi^2}{g^2} |\nu| \Leftrightarrow \nu = \frac{g^2}{16\pi^2} \int d^4 x_E \text{Tr} \left(F_E^{\mu\nu} \tilde{F}_{\mu\nu E} \right)$$

なので、

$$\boxed{S_E \geq \frac{8\pi^2}{g^2} |\nu|}$$

がわかる。等号は、

$$F_{\mu\nu E}^a = \pm \tilde{F}_{\mu\nu E}^a$$

のときに成立する。言い換えると、

- $\int d^4x_E \text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E})$ は、Action が有限であれば収束する物理量

であることが分かる。

そこで、「Action が有限」になることより

$$\int d^4x_E \text{Tr}(F_E^{\mu\nu} F_{\mu\nu E}) \sim r^4 (F_E^{\mu\nu})^2 \xrightarrow{r \rightarrow \infty} 0 \Rightarrow r^2 F_E^{\mu\nu} \xrightarrow{r \rightarrow \infty} 0$$

のおおまかな条件が分かる。つまり $F_E^{\mu\nu} \xrightarrow{r \rightarrow \infty} 0$ である。 $F_E^{\mu\nu} = 0$ の解は、「pure gauge 解」なので、

$$\begin{aligned} A_E^\mu(x) &\xrightarrow{r \rightarrow \infty} \frac{i}{g} U \partial^\mu U^\dagger \Leftarrow A'_E^\mu(x) = U A_E^\mu U^\dagger + \frac{i}{g} U \partial^\mu U^\dagger \\ X_\mu &\xrightarrow{r \rightarrow \infty} \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \\ &\Leftarrow X'_\mu = X_\mu - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} \frac{i}{g} \partial^\nu \text{Tr}(\partial^\rho U^\dagger \cdot U A_E^\sigma) + \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \end{aligned}$$

になる。この量は

$$X_\mu \rightarrow \frac{1}{r^3} \text{ as } A_E^\mu \left(\sim U \partial^\mu U^\dagger \right) \rightarrow \frac{1}{r} \text{ at infinity}$$

のもとで「表面積分」

$$\nu = \frac{g^2}{16\pi^2} \int d^4x_E \text{Tr}(F_E^{\mu\nu} \tilde{F}_{\mu\nu E}) = \frac{g^2}{4\pi^2} \int d^4x_E \partial^\mu X_\mu = \frac{g^2}{4\pi^2} \int d\sigma^\mu X_\mu$$

に寄与するので、

$$\begin{aligned} \nu &= \frac{g^2}{4\pi^2} \int d\sigma^\mu X_\mu = \frac{g^2}{4\pi^2} \int d\sigma^\mu \varepsilon_{\mu\nu\rho\sigma} \frac{1}{6} \left(\frac{1}{g} \right)^2 \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \\ &= \frac{1}{24\pi^2} \int d\sigma^\mu \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \text{ at infinity} \end{aligned}$$

と計算される。以上から

$$\boxed{\nu = \frac{1}{24\pi^2} \int d\sigma^\mu \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \text{ at infinity}}$$

である。

第三節 : SU(2) 解 ($\nu=1$)

ここで、 U として、

$$U = V_4 + i\tau V = \tau^\mu V_\mu, \quad U^\dagger = V_4 - i\tau V = \tau^{\dagger\mu} V_\mu \quad \text{with } \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1)$$

$$U = \begin{pmatrix} V_4 + iV_3 & iV_1 + V_2 \\ iV_1 - V_2 & V_4 - iV_3 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} V_4 - iV_3 & -iV_1 - V_2 \\ -iV_1 + V_2 & V_4 + iV_3 \end{pmatrix}$$

とすると、¹

$$\begin{aligned} \tau^\mu \tau^{\dagger\nu} &= \delta^{\mu\nu} + i\tau^{\mu\nu} \Leftarrow \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1) \\ \tau^4 \tau^{\dagger 4} &= 1, \tau^i \tau^{\dagger i} = (i\tau)^i (-i\tau)^i = 1 \quad (i: \text{not summed}) \\ \tau^4 \tau^{\dagger i} &= (-i\tau) = -i\tau^i, \tau^i \tau^{\dagger 4} = i\tau^i, \tau^4 \tau^{\dagger i} + \tau^i \tau^{\dagger 4} = 0 \\ \tau^i \tau^{\dagger j} &= (i\tau)^i (-i\tau)^j = \tau^i \tau^j = i\varepsilon^{ijk} \tau^k, \tau^i \tau^{\dagger j} + \tau^j \tau^{\dagger i} = 0 \quad (i \neq j) \end{aligned}$$

なので、まとめて、

$$\begin{aligned} \tau^\mu \tau^{\dagger\nu} + \tau^\nu \tau^{\dagger\mu} &= 2\delta^{\mu\nu}, \quad \tau^{\dagger\mu} \tau^\nu + \tau^{\dagger\nu} \tau^\mu = 2\delta^{\mu\nu} \\ \tau^\mu \tau^{\dagger\nu} - \tau^\nu \tau^{\dagger\mu} &= 2i\tau^{\mu\nu} = \begin{cases} \tau^4 \tau^{\dagger 4} - \tau^4 \tau^{\dagger 4} = 0 \\ \tau^4 \tau^{\dagger i} - \tau^i \tau^{\dagger 4} = (-i\tau^i) - i\tau^i = -2i\tau^i, \quad \tau^i \tau^{\dagger 4} - \tau^4 \tau^{\dagger i} = i\tau^i - (-i\tau^i) = 2i\tau^i \\ \tau^i \tau^{\dagger j} - \tau^j \tau^{\dagger i} = i\varepsilon^{ijk} \tau^k - i\varepsilon^{jik} \tau^k = 2i\varepsilon^{ijk} \tau^k \end{cases} \\ \tau^{\dagger\mu} \tau^\nu - \tau^{\dagger\nu} \tau^\mu &= 2i\bar{\tau}^{\mu\nu} = \begin{cases} \tau^{\dagger 4} \tau^4 - \tau^{\dagger 4} \tau^4 = 0 \\ \tau^{\dagger 4} \tau^i - \tau^{\dagger i} \tau^4 = i\tau^i - (-i\tau^i) = 2i\tau^i, \quad \tau^{\dagger i} \tau^4 - \tau^{\dagger 4} \tau^i = (-i\tau^i) - i\tau^i = -2i\tau^i \\ \tau^{\dagger i} \tau^j - \tau^{\dagger j} \tau^i = i\varepsilon^{ijk} \tau^k - i\varepsilon^{jik} \tau^k = 2i\varepsilon^{ijk} \tau^k \end{cases} \\ \frac{1}{2!} \varepsilon_{\mu\nu\rho\sigma} \tau^{\rho\sigma} &= \begin{cases} ijk4 : \tau^{k4} = \tau^k = \tau^{ij} \\ i4jk : \tau^{jk} = \varepsilon^{jki} \tau^i = \tau^i = \tau^{i4} \end{cases} = \tau^{\mu\nu}, \quad \frac{1}{2!} \varepsilon_{\mu\nu\rho\sigma} \bar{\tau}^{\rho\sigma} = \begin{cases} ijk4 : \bar{\tau}^{k4} = -\tau^k = -\bar{\tau}^{ij} \\ i4jk : \bar{\tau}^{jk} = \varepsilon^{jki} \tau^i = \tau^i = -\bar{\tau}^{i4} \end{cases} = -\bar{\tau}^{\mu\nu} \end{aligned}$$

である。以上から、

$$\boxed{\begin{aligned} \tau^\mu \tau^{\dagger\nu} + \tau^\nu \tau^{\dagger\mu} &= 2\delta^{\mu\nu}, \quad \tau^{\dagger\mu} \tau^\nu + \tau^{\dagger\nu} \tau^\mu = 2\delta^{\mu\nu} \\ \tau^\mu \tau^{\dagger\nu} - \tau^\nu \tau^{\dagger\mu} &= 2i\tau^{\mu\nu}, \quad \tau^{\dagger\mu} \tau^\nu - \tau^{\dagger\nu} \tau^\mu = 2i\bar{\tau}^{\mu\nu} \\ \tau^{\mu\nu} &= \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \tau^{\rho\sigma} \begin{cases} \tau^{4i} = -\tau^{i4} = -\tau^i \\ \tau^{ij} = -\tau^{ji} = \varepsilon^{ijk} \tau^k \end{cases}, \quad \bar{\tau}^{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \bar{\tau}^{\rho\sigma} \begin{cases} \bar{\tau}^{4i} = -\bar{\tau}^{i4} = \tau^i \\ \bar{\tau}^{ij} = -\bar{\tau}^{ji} = \varepsilon^{ijk} \tau^k \end{cases} \end{aligned}} \Leftarrow \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1)$$

従って、

$$1 = U^\dagger U = (\tau^\mu V_\mu)^\dagger (\tau^\nu V_\nu) = \tau^{\dagger\mu} \tau^\nu V_\mu V_\nu = \frac{\tau^{\dagger\mu} \tau^\nu + \tau^{\dagger\nu} \tau^\mu}{2} V_\mu V_\nu = \frac{2\delta^{\mu\nu}}{2} V_\mu V_\nu = V^\mu V_\mu = V^2 + V_4^2 (= UU^\dagger)$$

なので、

$$\boxed{V^2 + V_4^2 = 1}$$

¹ ここでは Euclid 計量の $\mu=0$ 成分を、 $\mu=4$ 成分として取り扱う。

になる。これを利用すると・・・

$$\begin{aligned}\partial^\nu U &= \partial^\nu (\tau^\alpha V_\alpha) = \tau^\alpha \partial^\nu V_\alpha, \partial^\nu U^\dagger = \partial^\nu (\tau^\alpha V_\alpha)^\dagger = \tau^{\dagger\alpha} \partial^\nu V_\alpha \\ U \partial^\mu U^\dagger &= \tau^\alpha V_\alpha \partial^\mu (\tau^\beta V_\beta)^\dagger = \tau^\alpha \tau^{\dagger\beta} V_\alpha \partial^\mu V_\beta = (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta\end{aligned}$$

$$A_E^\mu(x) \xrightarrow{r \rightarrow \infty} \frac{i}{g} U \partial^\mu U^\dagger = \frac{i}{g} (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta$$

を用いて、

$$\begin{aligned}\text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= \text{Tr}(U(-U^\dagger \partial^\nu U) \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) = -\text{Tr}(\partial^\nu U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \\ &= -\text{Tr}(U \partial^\sigma U^\dagger \partial^\nu U \partial^\rho U^\dagger) = -\text{Tr}(\tau^\alpha V_\alpha \tau^{\dagger\beta} \partial^\sigma V_\beta \tau^\gamma \partial^\nu V_\gamma \tau^{\dagger\delta} \partial^\rho V_\delta) = -\text{Tr}(\tau^\alpha \tau^{\dagger\beta} \tau^\gamma \tau^{\dagger\delta}) V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta \\ \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= -\text{Tr}(\tau^\alpha \tau^{\dagger\beta} \tau^\gamma \tau^{\dagger\delta})(\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\ &= -\text{Tr}\left(\tau^\alpha \tau^{\dagger\beta} \frac{\tau^\gamma \tau^{\dagger\delta} - \tau^\delta \tau^{\dagger\gamma}}{2}\right)(\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \Leftarrow \tau^\mu \tau^{\dagger\nu} - \tau^\nu \tau^{\dagger\mu} = 2i\tau^{\mu\nu} \\ &= -\text{Tr}\left(\tau^\alpha \tau^{\dagger\beta} \frac{2i\tau^{\gamma\delta}}{2}\right)(\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) = -i\text{Tr}(\tau^\alpha \tau^{\dagger\beta} \tau^{\gamma\delta})(\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\ &\Leftarrow \text{Tr}(\tau^\alpha \tau^{\dagger\beta} \tau^{\gamma\delta}) = \text{Tr}((\delta^{\alpha\beta} + i\tau^{\alpha\beta}) \tau^{\gamma\delta}) = \delta^{\alpha\beta} \text{Tr}(\tau^{\gamma\delta}) + i\text{Tr}(\tau^{\alpha\beta} \tau^{\gamma\delta}) = i\text{Tr}(\tau^{\alpha\beta} \tau^{\gamma\delta}) \\ (\alpha\beta\gamma\delta) &= (4ijk) : \text{Tr}(\tau^{\alpha\beta} \tau^{\gamma\delta}) = \text{Tr}(\tau^{4i} \tau^{jk}) = \text{Tr}(-\tau^i \varepsilon^{jk\ell} \tau^\ell) = -2\varepsilon^{jk\ell} \delta^{i\ell} = -2\varepsilon^{jki} = -2\varepsilon^{ijk} \\ \Rightarrow \text{Tr}(\tau^{\alpha\beta} \tau^{\gamma\delta}) &= 2\varepsilon^{\alpha\beta\gamma\delta} : \varepsilon^{4ijk} = -\varepsilon^{ijk4} = -\varepsilon^{ijk} \\ \Rightarrow \text{Tr}(\tau^\alpha \tau^{\dagger\beta} \tau^{\gamma\delta}) &= i\text{Tr}(\tau^{\alpha\beta} \tau^{\gamma\delta}) = i(2\varepsilon^{\alpha\beta\gamma\delta}) = 2i\varepsilon^{\alpha\beta\gamma\delta} \\ &= -i(2i\varepsilon^{\alpha\beta\gamma\delta})(\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) = 2\varepsilon^{\alpha\beta\gamma\delta} (\varepsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\ \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= 2\varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta)\end{aligned}$$

従って、

$$\begin{aligned}v &= \frac{1}{24\pi^2} \int d\sigma^\mu \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) = \frac{1}{12\pi^2} \int d\sigma^\mu \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\ &= \frac{1}{12\pi^2} \int d\sigma^\mu \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \stackrel{\text{付け直し}}{=} \frac{1}{12\pi^2} \int d\sigma^\mu \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta)\end{aligned}$$

なので、

$$v = \frac{1}{12\pi^2} \int d\sigma^\mu \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta)$$

である。一番簡単な U としては・・・

$$U = V_4 + i\tau V = \tau^\mu V_\mu \text{ with } V_\mu = \frac{x_\mu}{r} (\equiv \hat{x}_\mu) \quad (r = \sqrt{x^2})$$

であるので、

$$\begin{aligned}\partial^\mu V_\alpha &= \partial^\mu \left(\frac{x_\alpha}{r} \right) = \frac{\partial^\mu x_\alpha}{r} - \partial^\mu r \frac{x_\alpha}{r^2} = \frac{\delta_\alpha^\mu}{r} - \partial^\mu r \frac{x_\alpha}{r^2} \Leftarrow \partial^\mu r = \partial^\mu \sqrt{x^2} = \frac{2x^\alpha \partial^\mu x_\alpha}{2\sqrt{x^2}} = \frac{2x^\alpha \delta_\alpha^\mu}{2\sqrt{x^2}} = \frac{x^\mu}{r} \\ &= \frac{\delta_\alpha^\mu}{r} - \frac{x^\mu}{r} \frac{x_\alpha}{r^2} = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right)\end{aligned}$$

より

$$\partial^\mu V_\alpha = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right), \quad \partial^\mu U = \tau^\alpha \partial^\mu V_\alpha, \quad \partial^\mu U^\dagger = \tau^{\dagger\alpha} \partial^\mu V_\alpha$$

を使用すると、

$$\begin{aligned}U \partial^\mu U^\dagger &= (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta = (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) \\ &= \frac{x_\alpha}{r} \frac{1}{r} \delta^{\alpha\beta} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) + i\tau^{\alpha\beta} \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) \\ &= \frac{1}{r^2} \left(\cancel{x^\mu} \cancel{x^\mu} \frac{x^2}{r^2} \right) + i \frac{\tau^{\alpha\mu} x_\alpha}{r^2} - i \cancel{x_\alpha} \cancel{\tau^{\alpha\beta}} \cancel{x_\beta} \frac{x^\mu}{r^4} = i \frac{\tau^{\alpha\mu} x_\alpha}{r^2} = -i \frac{\tau^{\mu\nu} x_\nu}{r^2}\end{aligned}$$

$$A_E^\mu(x) \xrightarrow[r \rightarrow \infty]{i}{g} U \partial^\mu U^\dagger = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2}$$

従って、

$$X_\mu \rightarrow \frac{1}{r^3} \text{ as } A_E^\mu = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} \xrightarrow[r \rightarrow \infty]{|x^\mu| \rightarrow r} \frac{1}{r}$$

になるので、

$$\nu = \frac{g^2}{4\pi^2} \int d\sigma^\mu X_\mu \left(\sim r^3 \frac{1}{r^3} \text{ as } r \rightarrow \infty \right) \neq 0$$

を与える。実際、

$$\begin{aligned}\nu &= \frac{1}{12\pi^2} \int d\sigma^\mu \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right) \stackrel{\text{超平面}}{=} \frac{1}{12\pi^2} \int d\mathbf{S} \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} \hat{x}^\mu \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right) \Big|_{r \rightarrow \infty} \\ &= \frac{1}{12\pi^2} \int d\mathbf{S} \varepsilon^{\alpha\beta\gamma\delta} \hat{x}^\mu \varepsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right) \Big|_{r \rightarrow \infty} = \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \frac{1}{12\pi^2} \int d\mathbf{S} \hat{x}^\mu V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \Big|_{r \rightarrow \infty} \\ &\stackrel{\hat{x}^\mu = x^\mu/r}{=} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \frac{1}{12\pi^2} \int d\mathbf{S} \frac{x^\mu}{r} \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\nu - \frac{x^\nu x_\beta}{r^2} \right) \frac{1}{r} \left(\delta_\gamma^\rho - \frac{x^\rho x_\gamma}{r^2} \right) \frac{1}{r} \left(\delta_\delta^\sigma - \frac{x^\sigma x_\delta}{r^2} \right) \Big|_{r \rightarrow \infty} \\ &= \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \frac{1}{12\pi^2} \int d\mathbf{S} \frac{x^\mu}{r} \frac{x_\alpha}{r} \frac{1}{r} \delta_\beta^\nu \frac{1}{r} \delta_\gamma^\rho \frac{1}{r} \delta_\delta^\sigma \Big|_{r \rightarrow \infty} = \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \frac{1}{12\pi^2} \int d\mathbf{S} \hat{x}^\mu \frac{x_\alpha}{r^4} \Big|_{r \rightarrow \infty} \\ &\Leftarrow \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} = 3! \delta_\mu^\alpha\end{aligned}$$

$$\begin{aligned}
&= 3! \delta_\mu^\alpha \frac{1}{12\pi^2} \int d\mathbf{S} \hat{x}^\mu \frac{x_\alpha}{r^4} \Big|_{r \rightarrow \infty} = \frac{1}{2\pi^2} \int d\mathbf{S} \hat{x}^\mu \frac{x_\mu}{r^4} \Big|_{r \rightarrow \infty} = \frac{1}{2\pi^2} \int d\mathbf{S} \frac{r^2}{r} \frac{1}{r^4} \Big|_{r \rightarrow \infty} \\
&\Leftarrow \int d\mathbf{S} = 2\pi^2 r^3 \\
&= \frac{1}{2\pi^2} (2\pi^2 r^3) \frac{1}{r^3} \Big|_{r \rightarrow \infty} = 1
\end{aligned}$$

従って、

$$\boxed{\nu = 1 \text{ for } U = V_4 + i\tau V = \tau^\mu V_\mu \text{ with } V_\mu = \frac{x_\mu}{r} \quad (r = \sqrt{x^2})} \Leftarrow \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1)$$

がわかった。

第四節 : SU(2)解 ($\nu=1$)

今度は、 U として、 $\nu=1$ の場合の U^\dagger を用いる。従って、

$$\begin{aligned}
U &= \begin{pmatrix} V_4 - iV_3 & -iV_1 - V_2 \\ -iV_1 + V_2 & V_4 + iV_3 \end{pmatrix} = V_4 - i\tau V = \tau^{\dagger\mu} V_\mu \quad \text{with } \tau^{\dagger\mu} = (-i\tau, 1) \\
U^\dagger &= \begin{pmatrix} V_4 + iV_3 & iV_1 + V_2 \\ iV_1 - V_2 & V_4 - iV_3 \end{pmatrix} = V_4 + i\tau V = \tau^\mu V_\mu \quad \text{with } \tau^\mu = (i\tau, 1)
\end{aligned}$$

とすると、

$$\begin{aligned}
\text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= \text{Tr}(U(-U^\dagger \partial^\nu U) \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) = -\text{Tr}(\partial^\nu U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) \\
&= -\text{Tr}(U \partial^\sigma U^\dagger \partial^\nu U \partial^\rho U^\dagger) = -\text{Tr}(\tau^{\dagger\alpha} V_\alpha \tau^\beta \partial^\sigma V_\beta \tau^{\dagger\gamma} \partial^\nu V_\gamma \tau^\delta \partial^\rho V_\delta) = -\text{Tr}(\tau^{\dagger\alpha} \tau^\beta \tau^{\dagger\gamma} \tau^\delta) V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta \\
\epsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= -\text{Tr}(\tau^{\dagger\alpha} \tau^\beta \tau^{\dagger\gamma} \tau^\delta) (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\
&= -\text{Tr}\left(\tau^{\dagger\alpha} \tau^\beta \frac{\tau^{\dagger\gamma} \tau^\delta - \tau^{\dagger\delta} \tau^\gamma}{2}\right) (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \Leftarrow \tau^{\dagger\gamma} \tau^\delta - \tau^{\dagger\delta} \tau^\gamma = 2i\bar{\tau}^{\mu\nu} \\
&= -\text{Tr}\left(\tau^{\dagger\alpha} \tau^\beta \frac{2i\bar{\tau}^{\gamma\delta}}{2}\right) (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) = -i\text{Tr}(\tau^{\dagger\alpha} \tau^\beta \bar{\tau}^{\gamma\delta}) (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\
&\Leftarrow \text{Tr}(\tau^{\dagger\alpha} \tau^\beta \bar{\tau}^{\gamma\delta}) = \text{Tr}((\delta^{\alpha\beta} + i\bar{\tau}^{\alpha\beta}) \bar{\tau}^{\gamma\delta}) = \delta^{\alpha\beta} \text{Tr}(\bar{\tau}^{\gamma\delta}) + i\text{Tr}(\bar{\tau}^{\alpha\beta} \bar{\tau}^{\gamma\delta}) = i\text{Tr}(\bar{\tau}^{\alpha\beta} \bar{\tau}^{\gamma\delta}) \\
(\alpha\beta\gamma\delta) = (4ijk) : \text{Tr}(\bar{\tau}^{\alpha\beta} \bar{\tau}^{\gamma\delta}) &= \text{Tr}(\bar{\tau}^{4i} \bar{\tau}^{jk}) = \text{Tr}(\tau^i \epsilon^{jk\ell} \tau^\ell) = 2\epsilon^{jk\ell} \delta^{i\ell} = 2\epsilon^{jki} = 2\epsilon^{ijk} \\
\Rightarrow \text{Tr}(\bar{\tau}^{\alpha\beta} \bar{\tau}^{\gamma\delta}) &= -2\epsilon^{\alpha\beta\gamma\delta} : \epsilon^{4ijk} = -\epsilon^{ijk4} = -\epsilon^{ijkl} \\
\Rightarrow \text{Tr}(\tau^{\dagger\alpha} \tau^\beta \tau^{\gamma\delta}) &= i\text{Tr}(\bar{\tau}^{\alpha\beta} \bar{\tau}^{\gamma\delta}) = i(-2\epsilon^{\alpha\beta\gamma\delta}) = -2i\epsilon^{\alpha\beta\gamma\delta} \\
&= -i(-2i\epsilon^{\alpha\beta\gamma\delta}) (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) = -2\epsilon^{\alpha\beta\gamma\delta} (\epsilon_{\mu\nu\rho\sigma} V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta) \\
\epsilon_{\mu\nu\rho\sigma} \text{Tr}(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger) &= -2\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta)
\end{aligned}$$

従って、

$$\nu = \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) = -\frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta \right)$$

$$= -\frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\sigma V_\beta \partial^\nu V_\gamma \partial^\rho V_\delta \right) \stackrel{\mu\sigma\nu\rho}{\text{付け直し}} = -\frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right)$$

なので、

$$\boxed{\nu = -\frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right)}$$

である。一番簡単な U として、前と同様に

$$V_\mu = \frac{x_\mu}{r} (\equiv \hat{x}_\mu) \quad (r = \sqrt{x^2})$$

を使えば、

$$\begin{aligned} U \partial^\mu U^\dagger &= (\delta^{\alpha\beta} + i\bar{\tau}^{\alpha\beta}) V_\alpha \partial^\mu V_\beta = (\delta^{\alpha\beta} + i\bar{\tau}^{\alpha\beta}) \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) \\ &= \frac{x_\alpha}{r} \frac{1}{r} \delta^{\alpha\beta} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) + i\bar{\tau}^{\alpha\beta} \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\mu - \frac{x^\mu x_\beta}{r^2} \right) \\ &= \frac{1}{r^2} \left(\cancel{x^\mu} - \frac{\cancel{x^2} \cancel{x^\mu}}{r^2} \right) + i \frac{\bar{\tau}^{\alpha\mu} x_\alpha}{r^2} - i \cancel{x_\alpha} \cancel{\bar{\tau}^{\alpha\beta}} \cancel{x_\beta} \frac{x^\mu}{r^4} = i \frac{\bar{\tau}^{\alpha\mu} x_\alpha}{r^2} = -i \frac{\bar{\tau}^{\mu\nu} x_\nu}{r^2} \end{aligned}$$

$$\boxed{A_E^\mu(x) \xrightarrow{r \rightarrow \infty} \frac{i}{g} U \partial^\mu U^\dagger = \frac{1}{g} \frac{\bar{\tau}^{\mu\nu} x_\nu}{r^2}}$$

そして、

$$\nu = -\frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \left(V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta \right) = -1$$

従って、

$$\boxed{\nu = -1 \text{ for } U = V_4 - i\tau V = \bar{\tau}^\mu V_\mu \text{ with } V_\mu = \frac{x_\mu}{r} \quad (r = \sqrt{x^2})} \Leftarrow \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1)$$

がわかった。

第五節 : SU(2) 解 ($|\nu| > 1$)

$$\begin{aligned} \nu &= \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) \\ U &= u^n \quad \text{with } u = \tau^\mu \frac{x_\mu}{r}, \quad \partial^\nu u = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right) \Leftarrow \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(u \partial^\nu u^\dagger \cdot u \partial^\rho u^\dagger \cdot u \partial^\sigma u^\dagger \right) = 1 \\ U \partial^\mu U^\dagger &= u^n \partial^\mu u^{\dagger n} = uu^{n-1} \partial^\mu \left(u^{\dagger n-1} u^\dagger \right) = u \left(u^{n-1} \partial^\mu u^{\dagger n-1} \right) u^\dagger + u \partial^\mu u^\dagger \end{aligned}$$

そこで、

$$\begin{aligned}
\nu(n) &\equiv \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u^n \partial^\nu u^{\dagger n} \right) \left(u^n \partial^\rho u^{\dagger n} \right) \left(u^n \partial^\sigma u^{\dagger n} \right) \right) \\
&= \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u^{n-1} \partial^\nu u^{\dagger n-1} \right) \left(u^{n-1} \partial^\rho u^{\dagger n-1} \right) \left(u^{n-1} \partial^\sigma u^{\dagger n-1} \right) \right) \\
&\quad + \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u \partial^\nu u^\dagger \right) \left(u \partial^\rho u^\dagger \right) \left(u \partial^\sigma u^\dagger \right) \right) - \frac{1}{8\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{Tr} \left(u^n \partial^\rho u^{\dagger n-1} \partial^\sigma u^\dagger \right) \\
&= \nu(n-1) + \nu(1) - \frac{1}{8\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{Tr} \left(u^n \partial^\rho u^{\dagger n-1} \partial^\sigma u^\dagger \right)
\end{aligned}$$

ここで、

$$\begin{aligned}
\frac{1}{8\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{Tr} \left(u^n \partial^\rho u^{\dagger n-1} \partial^\sigma u^\dagger \right) &\stackrel{t=-\text{定}}{=} \frac{1}{8\pi^2} 3! \int d\sigma^0 (-\nabla_x) \text{Tr} \left(u^n (-\nabla_y) u^{\dagger n-1} (-\nabla_z) u^\dagger \right) \\
&= -\frac{1}{8\pi^2} 3! \int dx \nabla_x \text{Tr} \left(u^n \nabla_y u^{\dagger n-1} \nabla_z u^\dagger \right) = -\frac{1}{8\pi^2} 3! \int dx dy \text{Tr} \left(u^n \nabla_y u^{\dagger n-1} \nabla_z u^\dagger \right) \Big|_{x=\pm\infty} = 0
\end{aligned}$$

なので、

$$\begin{aligned}
\nu(n) &\equiv \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u^{n-1} \partial^\nu u^{\dagger n-1} \right) \left(u^{n-1} \partial^\rho u^{\dagger n-1} \right) \left(u^{n-1} \partial^\sigma u^{\dagger n-1} \right) \right) \\
&= \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u^{n-1} \partial^\nu u^{\dagger n-1} \right) \left(u^{n-1} \partial^\rho u^{\dagger n-1} \right) \left(u^{n-1} \partial^\sigma u^{\dagger n-1} \right) \right) \\
&\quad + \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(\left(u \partial^\nu u^\dagger \right) \left(u \partial^\rho u^\dagger \right) \left(u \partial^\sigma u^\dagger \right) \right) - \frac{1}{8\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{Tr} \left(u^n \partial^\rho u^{\dagger n-1} \partial^\sigma u^\dagger \right) \\
&= \nu(n-1) + \nu(1) - \frac{1}{8\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{Tr} \left(u^n \partial^\rho u^{\dagger n-1} \partial^\sigma u^\dagger \right)
\end{aligned}$$

$$\nu(n) = \nu(n-1) + \nu(1)$$

を得る。 $n=2$ では、 $\nu(2) = \nu(1) + \nu(1) = 2\nu(1) = 2$ がわかるので、 $\nu(n) = n$ になる。 以上から、

$$\nu = \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(U \partial^\nu U^\dagger \cdot U \partial^\rho U^\dagger \cdot U \partial^\sigma U^\dagger \right) = n \Leftarrow U = u^n \quad \text{with } u = \tau^\mu \frac{x_\mu}{r}, \quad \partial^\nu u = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right)$$

である。 同様にして、

$$\nu = \frac{1}{24\pi^2} \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left(U^\dagger \partial^\nu U \cdot U^\dagger \partial^\rho U \cdot U^\dagger \partial^\sigma U \right) = -n \Leftarrow U = u^n \quad \text{with } u = \tau^\mu \frac{x_\mu}{r}, \quad \partial^\nu u = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right)$$

である。

第六節：有限エネルギー解

$\nu = \pm 1$ を与える pure gauge 解は、もちろん、

$$F_E^{\mu\nu} = 0$$

である。 そこで、有限エネルギー解を、

$$A_E^\mu(x) = \frac{i}{g} U \partial^\mu U^\dagger f(r^2) = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} f(r^2) \equiv a^\mu(x) f(r^2) \left(a^\mu = \frac{i}{g} (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta \right)$$

とし、

$$f(0) = 0 \quad (\text{regular at } 0) \Leftarrow A_E^\mu(0) < \infty$$

$$f(\infty) = 1 \quad (\text{pure gauge at } \infty) \Leftarrow A_E^\mu(x) \xrightarrow[r \rightarrow \infty]{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} \quad \left(A_E^\mu(x) \xrightarrow[r \rightarrow \infty]{i}{g} U \partial^\mu U^\dagger = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} \right)$$

を課す。この解を使って、

$$\begin{aligned} F_E^{\mu\nu} &= \partial^\mu A_E^\nu - \partial^\nu A_E^\mu - ig [A_E^\mu, A_E^\nu] = \partial^\mu (a^\nu f) - \partial^\nu (a^\mu f) - ig [a^\mu f, a^\nu f] \\ &= \left(\underbrace{\partial^\mu a^\nu - \partial^\nu a^\mu}_{-ig [a^\mu, a^\nu]} \right) f + a^\nu \partial^\mu f - a^\mu \partial^\nu f - ig [a^\mu, a^\nu] (f^2 - f) \\ &= a^\nu \partial^\mu f - a^\mu \partial^\nu f - ig [a^\mu, a^\nu] (f^2 - f) \end{aligned}$$

$$\begin{aligned}
& \Leftarrow [a^\mu, a^\nu] = \left[\frac{i}{g} U \partial^\mu U^\dagger, \frac{i}{g} U \partial^\nu U^\dagger \right] = \left(\frac{i}{g} \right)^2 (U \partial^\mu U^\dagger \cdot U \partial^\nu U^\dagger - U \partial^\nu U^\dagger \cdot U \partial^\mu U^\dagger) \\
& = -\frac{1}{g^2} (-\partial^\mu U \cdot \partial^\nu U^\dagger + \partial^\nu U \partial^\mu U^\dagger) = \frac{1}{g^2} (\partial^\mu U \cdot \partial^\nu U^\dagger - \partial^\nu U \partial^\mu U^\dagger) \\
& \quad \leftarrow \partial^\mu U = \tau^\alpha \partial^\mu V_\alpha, \quad \partial^\mu U^\dagger = \tau^{\dagger\alpha} \partial^\mu V_\alpha \\
& = \frac{1}{g^2} (\tau^\alpha \partial^\mu V_\alpha \cdot \tau^{\dagger\beta} \partial^\nu V_\beta - \tau^\alpha \partial^\nu V_\alpha \cdot \tau^{\dagger\beta} \partial^\mu V_\beta) = \frac{1}{g^2} \tau^\alpha \tau^{\dagger\beta} (\partial^\mu V_\alpha \partial^\nu V_\beta - \partial^\nu V_\alpha \partial^\mu V_\beta) \\
& = \frac{1}{g^2} (\tau^\alpha \tau^{\dagger\beta} - \tau^\beta \tau^{\dagger\alpha}) \partial^\mu V_\alpha \partial^\nu V_\beta = \frac{1}{g^2} 2i \tau^{\alpha\beta} \partial^\mu V_\alpha \partial^\nu V_\beta \\
& \Leftarrow a^\mu = \frac{i}{g} (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta \\
& = \frac{i}{g} [\boxed{(\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\nu V_\beta \cdot \partial^\mu f}] - \frac{i}{g} (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\mu V_\beta \cdot \partial^\nu f - ig \frac{1}{g^2} 2i [\boxed{\tau^{\alpha\beta} \partial^\mu V_\alpha \partial^\nu V_\beta}] (f^2 - f) \\
& \Leftarrow V_\alpha = \frac{x_\alpha}{r}, \quad \partial^\mu V_\alpha = \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right), \quad \partial^\mu f = (\partial^\mu r^2) f' = (\partial^\mu x^2) f' = 2x^\mu f' \\
& \Leftarrow [\boxed{(\delta^{\alpha\beta} + i\tau^{\alpha\beta}) V_\alpha \partial^\nu V_\beta \cdot \partial^\mu f}] = (\delta^{\alpha\beta} + i\tau^{\alpha\beta}) \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\nu - \frac{x^\nu x_\beta}{r^2} \right) \cdot 2x^\mu f' \\
& = \delta^{\alpha\beta} \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\nu - \frac{x^\nu x_\beta}{r^2} \right) \cdot 2x^\mu f' + i\tau^{\alpha\beta} \frac{x_\alpha}{r} \frac{1}{r} \left(\delta_\beta^\nu - \frac{x^\nu x_\beta}{r^2} \right) 2x^\mu f' \\
& = \frac{1}{r^2} \left(\cancel{x^\nu - \frac{x^\nu x_\beta}{r^2}} \right) \cdot 2x^\mu f' + i \frac{1}{r^2} \left(x_\alpha \tau^{\alpha\nu} - \cancel{\frac{x^\nu x_\alpha \tau^{\alpha\beta} x_\beta}{r^2}} \right) 2x^\mu f' = 2i \frac{1}{r^2} x_\alpha \tau^{\alpha\nu} x^\mu f' \\
& \Leftarrow [\boxed{\tau^{\alpha\beta} \partial^\mu V_\alpha \partial^\nu V_\beta}] = \tau^{\alpha\beta} \frac{1}{r} \left(\delta_\alpha^\mu - \frac{x^\mu x_\alpha}{r^2} \right) \frac{1}{r} \left(\delta_\beta^\nu - \frac{x^\nu x_\beta}{r^2} \right) \\
& = \frac{1}{r^2} \left(\tau^{\alpha\beta} \delta_\alpha^\mu \delta_\beta^\nu - \tau^{\alpha\beta} \frac{x^\mu x_\alpha}{r^2} \delta_\beta^\nu - \tau^{\alpha\beta} \delta_\alpha^\mu \frac{x^\nu x_\beta}{r^2} + \cancel{\frac{x^\mu x_\alpha \tau^{\alpha\beta} x_\beta}{r^2}} \right) \\
& = \frac{1}{r^2} \left(\tau^{\mu\nu} - \frac{x_\alpha \tau^{\alpha\nu} x^\mu}{r^2} - \frac{x_\beta \tau^{\mu\beta} x^\nu}{r^2} \right) = \frac{1}{r^2} \left(\tau^{\mu\nu} - \frac{x_\alpha \tau^{\alpha\nu} x^\mu}{r^2} + \frac{x_\beta \tau^{\beta\mu} x^\nu}{r^2} \right) \\
& = \frac{i}{g} 2i \frac{1}{r^2} x_\alpha \tau^{\alpha\nu} x^\mu f' - \frac{i}{g} 2i \frac{1}{r^2} x_\alpha \tau^{\alpha\mu} x^\nu f' - ig \frac{1}{g^2} 2i \frac{1}{r^2} \left(\tau^{\mu\nu} - \frac{x_\alpha \tau^{\alpha\nu} x^\mu}{r^2} + \frac{x_\beta \tau^{\beta\mu} x^\nu}{r^2} \right) (f^2 - f) \\
& = -\frac{2}{g} \frac{1}{r^2} (x_\alpha \tau^{\alpha\nu} x^\mu - x_\alpha \tau^{\alpha\mu} x^\nu) f' + \frac{2}{g} \frac{1}{r^2} \left(\tau^{\mu\nu} - \frac{x_\alpha \tau^{\alpha\nu} x^\mu}{r^2} + \frac{x_\beta \tau^{\beta\mu} x^\nu}{r^2} \right) (f^2 - f) \\
& = \frac{2}{g} \left[\frac{x_\alpha \tau^{\alpha\nu} x^\mu - x_\alpha \tau^{\alpha\mu} x^\nu}{r^2} \left(-\frac{f^2 - f}{r^2} - f' \right) + \frac{f^2 - f}{r^2} \tau^{\mu\nu} \right]
\end{aligned}$$

従って、

$$F_E^{\mu\nu} = \frac{2}{g} \left[\frac{x_\alpha \tau^{\alpha\nu} x^\mu - x_\alpha \tau^{\alpha\mu} x^\nu}{r^2} \left(\frac{f - f^2}{r^2} - f' \right) + \frac{f^2 - f}{r^2} \tau^{\mu\nu} \right]$$

である。ここで、

$$\tau^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tau^{\rho\sigma}$$

に注意すると、

$$F_E^{\mu\nu} = \frac{2}{g} \frac{f^2 - f}{r^2} \tau^{\mu\nu} \Leftrightarrow \tilde{F}_E^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma E} = \frac{2}{g} \frac{f^2 - f}{r^2} \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \tau^{\mu\nu} = \frac{2}{g} \frac{f^2 - f}{r^2} \tau^{\mu\nu} = F_E^{\mu\nu}$$

なので、

$$\frac{f - f^2}{r^2} - f' = 0$$

の解として f が求められる。

$$\begin{aligned} \frac{f - f^2}{r^2} - f' &= 0 \Rightarrow \frac{f - f^2}{r^2} - \frac{1}{r^2} \left[\left(\frac{r^2}{f} \right)' - \frac{1}{f} \right] (-f^2) = 0 \\ &\Rightarrow \frac{f - f^2}{r^2} + \frac{1}{r^2} \left[\left(\frac{r^2}{f} \right)' f^2 - f \right] = \frac{f^2}{r^2} \left[\left(\frac{r^2}{f} \right)' - 1 \right] = 0 \Rightarrow \left(\frac{r^2}{f} \right)' = 1 \\ &\Rightarrow \frac{r^2}{f} = r^2 + \text{cst} \Rightarrow f = \frac{r^2}{r^2 + \text{cst}} \Rightarrow f = \frac{r^2}{r^2 + \rho^2} \end{aligned}$$

従って、

$$f(r^2) = \frac{r^2}{r^2 + \rho^2} = 1 - \frac{\rho^2}{r^2 + \rho^2}, 1 - f(r^2) = \frac{\rho^2}{r^2 + \rho^2}$$

より

$$A_E^\mu(x) = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} f(r^2) = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2} \frac{r^2}{r^2 + \rho^2} = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2 + \rho^2}$$

である。また、

$$F_E^{\mu\nu} = \frac{2}{g} \frac{f^2 - f}{r^2} \tau^{\mu\nu} = \frac{2}{g} \frac{1}{r^2} \frac{r^2}{r^2 + \rho^2} \frac{\rho^2}{r^2 + \rho^2} \tau^{\mu\nu} = \frac{2}{g} \frac{\rho^2}{(r^2 + \rho^2)^2} \tau^{\mu\nu}$$

以上から、

$$\boxed{A_E^\mu = \frac{1}{g} \frac{\tau^{\mu\nu} x_\nu}{r^2 + \rho^2}, F_E^{\mu\nu} = \frac{2}{g} \frac{\rho^2}{(r^2 + \rho^2)^2} \tau^{\mu\nu}} \Leftarrow \begin{cases} \tau^\mu \tau^{\dagger\nu} - \tau^\nu \tau^{\dagger\mu} = 2i\tau^{\mu\nu}, \quad \tau^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tau^{\rho\sigma} \\ \tau^\mu = (i\tau, 1), \tau^{\dagger\mu} = (-i\tau, 1) \end{cases}$$

がわかる。

第七節 : Winding Number

Topological charge \mathcal{N} は、

$$\nu = \pm \frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta) \begin{cases} + \text{ for } U = V_4 + i\tau V = \tau^\mu V_\mu \\ - \text{ for } U = V_4 - i\tau V = \tau^{\dagger\mu} V_\mu \end{cases}$$

である。そこで、

$$d\sigma^\mu = d\mathbf{S} \hat{x}^\mu \left(\hat{x}^\mu = \frac{x^\mu}{r} \text{ with } r = \sqrt{x^2} \right)$$

とすれば、 $d\mathbf{S}$ は、 \hat{x}^μ に垂直な3つのベクトルで指定できる。4Dの回転により、

$$\hat{x}^\mu = (0, 0, 0, 1)$$

にできるので、

- $d\mathbf{S}$ は、 \hat{x}^4 に垂直な3つのベクトル $\mathbf{x}_{i=1,2,3}$ で指定

できる。従って、Jacobian (J)を用いて、

$$V(\mathbf{x}) \rightarrow \mathbf{x}$$

の変数変換により

$$\int d\mathbf{S}(V) f(V) = \int d\mathbf{S}(\mathbf{x}) J(\mathbf{x}) f(V(\mathbf{x}))$$

になる。ここに、

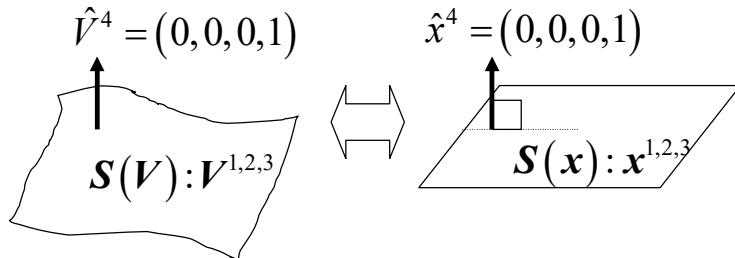
$$J(\mathbf{x}) = \begin{vmatrix} \frac{\partial \mathbf{V}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{V}_1}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{V}_1}{\partial \mathbf{x}_3} \\ \frac{\partial \mathbf{V}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{V}_2}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{V}_2}{\partial \mathbf{x}_3} \\ \frac{\partial \mathbf{V}_3}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{V}_3}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{V}_3}{\partial \mathbf{x}_3} \end{vmatrix} = \frac{1}{3!} \epsilon^{4mn\ell} \epsilon_{4ijk} \partial^i \mathbf{V}_m \partial^j \mathbf{V}_n \partial^k \mathbf{V}_\ell$$

である。これを共変形式にすると、

$$J(\mathbf{x}) = \frac{1}{3!} \epsilon^{4mn\ell} \epsilon_{4ijk} \partial^i \mathbf{V}_m \partial^j \mathbf{V}_n \partial^k \mathbf{V}_\ell \stackrel{V_4=1}{=} \frac{1}{3!} \hat{x}^\mu V_4 \epsilon^{4mn\ell} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \mathbf{V}_m \partial^\rho \mathbf{V}_n \partial^\sigma \mathbf{V}_\ell \stackrel{\text{共変化}}{=} \frac{1}{3!} \hat{x}^\mu V_\alpha \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \mathbf{V}_\beta \partial^\rho \mathbf{V}_\gamma \partial^\sigma \mathbf{V}_\delta$$

ここに、

V_4 は V に垂直にとる



である。従って、

$$\int dS(V)f(V) = \int dS(x) \frac{1}{3!} \hat{x}^\mu V_\alpha \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta f(V(x))$$

ここで、

$$f(V) = 1$$

とすると、

$$\int dS(V) = \frac{1}{6} \int dS(x) \hat{x}^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta) = \frac{1}{6} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta)$$

なので

$$\nu = \pm \frac{1}{12\pi^2} \int d\sigma^\mu \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} (V_\alpha \partial^\nu V_\beta \partial^\rho V_\gamma \partial^\sigma V_\delta) = \pm \frac{1}{2\pi^2} \int dS(V)$$

になる。4次元では、

$$4\text{次元の立体角} = 2\pi^2$$

なので、

$$\int dS(V) = 2\pi^2$$

である。従って、

$$\nu = \pm 1$$

になる。一般に、

- 実空間で1周するとき、 V 空間で n ($n=1,2,3,\dots$) 周できるので

$$\int dS(V) = 2\pi^2 n \quad (n=1,2,3,\dots)$$

であり、Topological charge は

$$\nu = \pm n \quad (n=1,2,3,\dots)$$

になる。