

第一章 CPT対称性 I

－例を使って（スピニ=0、1/2、1）－

第一節：空間反転（P変換） $x \rightarrow -x$

$\mathbf{p} \rightarrow -\mathbf{p}$ （運動量）、 $\mathbf{L} \rightarrow \mathbf{L}$ （角運動量 $\mathbf{x} \times \mathbf{p}$ ）、 $\mathbf{S} \rightarrow \mathbf{S}$ （スピニ）、 $h \rightarrow -h$ （ヘリシティ）
ここに、 $h = \mathbf{p} \cdot \mathbf{S} / |\mathbf{p}|$

スピニ0

粒子・反粒子の消滅・生成演算子を

粒子： $a(\mathbf{p}), a^\dagger(\mathbf{p})$ 、反粒子： $b(\mathbf{p}), b^\dagger(\mathbf{p})$

とする。交換関係を不変にするため、空間反転により

粒子： $[a(\mathbf{p}), a^\dagger(\mathbf{p})] = 1, [a(-\mathbf{p}), a^\dagger(-\mathbf{p})] = 1 \Rightarrow a(-\mathbf{p}) = \eta_p a(\mathbf{p}), a^\dagger(-\mathbf{p}) = \eta_p^* a^\dagger(\mathbf{p}) \Leftrightarrow |\eta_p|^2 = 1$

反粒子： $[b(\mathbf{p}), b^\dagger(\mathbf{p})] = 1, [b(-\mathbf{p}), b^\dagger(-\mathbf{p})] = 1 \Rightarrow b(-\mathbf{p}) = \eta'_p b(\mathbf{p}), b^\dagger(-\mathbf{p}) = \eta'_p b^\dagger(\mathbf{p}) \Leftrightarrow |\eta'_p|^2 = 1$

空間反転の演算子を \mathcal{U}_p とすると、

$$\mathcal{U}_p a(\mathbf{p}) \mathcal{U}_p^{-1} = a(-\mathbf{p}) = \eta_p a(\mathbf{p}), \mathcal{U}_p a^\dagger(\mathbf{p}) \mathcal{U}_p^{-1} = a^\dagger(-\mathbf{p}) = \eta_p^* a^\dagger(\mathbf{p}) \Leftrightarrow |\eta_p|^2 = 1$$

$$\mathcal{U}_p b(\mathbf{p}) \mathcal{U}_p^{-1} = b(-\mathbf{p}) = \eta'_p b(\mathbf{p}), \mathcal{U}_p b^\dagger(\mathbf{p}) \mathcal{U}_p^{-1} = b^\dagger(-\mathbf{p}) = \eta'_p b^\dagger(\mathbf{p}) \Leftrightarrow |\eta'_p|^2 = 1$$

一方、粒子・反粒子は一つの場を形成するので、 η_p, η'_p は、

$$\begin{aligned} \phi(x) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-ipx) + b^\dagger(\mathbf{p}) \exp(ipx)] = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-iEt + ipx) + b^\dagger(\mathbf{p}) \exp(iEt - ipx)] \\ \phi(t, -x) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-iEt + ip(-x)) + b^\dagger(\mathbf{p}) \exp(iEt - ip(-x))] \\ &= \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} [a(-\mathbf{p}) \exp(-iEt + i(-\mathbf{p})(-x)) + b^\dagger(-\mathbf{p}) \exp(iEt - i(-\mathbf{p})(-x))] \\ &= \int_{-\infty}^{\infty} (-)^3 \frac{d\mathbf{p}}{2E} [a(-\mathbf{p}) \exp(-iEt + ipx) + b^\dagger(-\mathbf{p}) \exp(iEt - ipx)] \\ &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_p a(\mathbf{p}) \exp(-ipx) + \eta'_p b^\dagger(\mathbf{p}) \exp(ipx)] \end{aligned}$$

これが元の場で書けるには、 $\eta'_p = \eta_p$ の必要がある。その結果、

$$\begin{aligned} \phi(t, -x) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_p a(\mathbf{p}) \exp(-ipx) + \eta'_p b^\dagger(\mathbf{p}) \exp(ipx)] = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_p a(\mathbf{p}) \exp(-ipx) + \eta_p b^\dagger(\mathbf{p}) \exp(ipx)] \\ &= \eta_p \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-ipx) + b^\dagger(\mathbf{p}) \exp(ipx)] = \eta_p \phi(t, x) \end{aligned}$$

従って、

$$\boxed{\mathcal{U}_P \phi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \phi(t, -\mathbf{x}) = \eta_p \phi(t, \mathbf{x}), \mathcal{U}_P \phi^\dagger(t, \mathbf{x}) \mathcal{U}_P^{-1} = \phi^\dagger(t, -\mathbf{x}) = \eta_p^* \phi^\dagger(t, \mathbf{x})}$$

$$a(-\mathbf{p}) = \eta_p a(\mathbf{p}), b(-\mathbf{p}) = \eta_p^* b(\mathbf{p}) \Leftarrow |\eta_p|^2 = 1 \text{ for } \eta_p = + (\text{スカラ一}), \eta_p = - (\text{擬スカラ一})$$

スピニン 1 (光子)

$$a_\mu(\mathbf{p}), a_\mu^\dagger(\mathbf{p})$$

質量の無い光子は

$$\text{ローレンツ条件 : } p^\mu a_\mu(\mathbf{p}) = 0, \text{ ゲージ変換 : } a'_\mu(\mathbf{p}) = a_\mu(\mathbf{p}) - \lambda p_\mu$$

の 2 つの拘束条件を持つ。ゲージ変換より、 $a'_0(\mathbf{p}) = a_0(\mathbf{p}) - \lambda p_0 = 0$ となる λ

$$\lambda = \frac{a_0(\mathbf{p})}{p_0}$$

をいつでも取ることができ、ゲージ不変性より $a_\mu(\mathbf{p})$ でも $a'_\mu(\mathbf{p})$ でも同じ物理であることが保証されるので、 $a_0(\mathbf{p}) = 0$ と設定してもよい。このとき、ローレンツ条件より、

$$p^\mu a_\mu(\mathbf{p}) = -\mathbf{p} \cdot \mathbf{a}(\mathbf{p}) = -\sum_{i=1,2,3} \mathbf{p}^i a^i(\mathbf{p}) = 0$$

である。4 つのローレンツベクトル $e_\mu^{(1,2,3,4)}(\mathbf{p})$:

$$e^{(1)\mu}(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e^{(2)\mu}(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e^{(3)\mu}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e^{(4)\mu}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } \begin{cases} \mu = 1 \\ \mu = 2 \\ \mu = 3 \\ \mu = 0 \end{cases}$$

$$\Rightarrow e_\mu^{(\lambda)} e^{(\lambda')*\mu} = g^{\lambda\lambda'} = \text{diag.}(-1, -1, -1, 1), \sum_{\lambda, \lambda'=1}^4 g^{\lambda\lambda'} e_\mu^{(\lambda)} e_\nu^{(\lambda')*} = g_{\mu\nu}$$

を用いると、対応する 4 つの生成・消滅演算子 $a^{(1,2,3,4)}(\mathbf{p}), a^{(1,2,3,4)\dagger}(\mathbf{p})$ を用いて

$$a^\mu(\mathbf{p}) = \sum_{\lambda=1}^4 e^{(\lambda)\mu}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) = \begin{pmatrix} a^{(1)}(\mathbf{p}) \\ a^{(2)}(\mathbf{p}) \\ a^{(3)}(\mathbf{p}) \\ a^{(4)}(\mathbf{p}) \end{pmatrix}, [a^{(\lambda)}(\mathbf{p}), a^{(\lambda')\dagger}(\mathbf{p}')] = -g^{\lambda\lambda'} \delta(\mathbf{p} - \mathbf{p}')$$

となる。交換関係は、

$$[a_\mu(\mathbf{p}), a_\nu^\dagger(\mathbf{p}')] = \left[\sum_{\lambda=1}^4 e_\mu^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}), \sum_{\lambda'=1}^4 e_\nu^{(\lambda')*}(\mathbf{p}') a^{(\lambda')\dagger}(\mathbf{p}') \right] = -\sum_{\lambda=1}^4 g^{\lambda\lambda'} e_\mu^{(\lambda)}(\mathbf{p}) e_\nu^{(\lambda')*}(\mathbf{p}') = -g_{\mu\nu} \delta(\mathbf{p} - \mathbf{p}')$$

で与えられる。このとき、

$$a_0(\mathbf{p}) \Rightarrow a^{(4)}(\mathbf{p}) = 0$$

$$\mathbf{p} \cdot \mathbf{a}(\mathbf{p}) = \sum_{\lambda=1}^4 \mathbf{p} \cdot \mathbf{e}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) = \sum_{\lambda=1}^3 \mathbf{p} \cdot \mathbf{e}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) = 0 \Rightarrow \mathbf{p} \cdot \mathbf{e}^{(\lambda)}(\mathbf{p}) = 0 \ (\lambda = 1, 2, 3)$$

と表される。ここで、射影演算子の

$$P^{ij} \equiv \delta^{ij} - \frac{\mathbf{p}^i \mathbf{p}^j}{|\mathbf{p}|^2} (= P^{ji})$$

より

$$\sum_{j=1,2,3} P^{ij} \mathbf{p}^j = \sum_{j=1,2,3} \left(\delta^{ij} - \frac{\mathbf{p}^i \mathbf{p}^j}{|\mathbf{p}|^2} \right) \mathbf{p}^j = \mathbf{p}^i - \frac{\mathbf{p}^i}{|\mathbf{p}|^2} \sum_{j=1,2,3} \mathbf{p}^j \mathbf{p}^j = \mathbf{p}^i - \frac{\mathbf{p}^i}{|\mathbf{p}|^2} |\mathbf{p}|^2 = 0$$

なので任意のベクトルを $\mathbf{n}^{(\lambda)}$ とすれば

$$\mathbf{e}^{(\lambda)}(\mathbf{p}) \equiv \sum_{i,j=1,2,3} P^{ij} \mathbf{n}^{(\lambda)} \Leftrightarrow \mathbf{e}^{(\lambda)i} \mathbf{p}^i = \mathbf{e}^{(\lambda)} \cdot \mathbf{p} = 0 \quad (\mathbf{e}^{(\lambda)} \text{と } \mathbf{p} \text{は常に直行})$$

がわかる。従って $\mathbf{p} \cdot \mathbf{e}^{(\lambda)}(\mathbf{p}) = 0$ を満たす $\mathbf{e}^{(\lambda)}(\mathbf{p})$ として

$$\mathbf{e}^{(\lambda)i}(\mathbf{p}) = \sum_{j=1,2,3} P^{ij} \mathbf{n}^{(\lambda)j} \quad (\text{for } \mathbf{n}^{(\lambda)})$$

とあらわすことができる。以上から、

$$a_\mu(\mathbf{p}) = \sum_{\lambda=1}^3 e_\mu^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \text{ with } \mathbf{e}^{(\lambda)i}(\mathbf{p}) = \sum_{j=1,2,3} P^{ij} \mathbf{n}^{(\lambda)j}, a^{(4)}(\mathbf{p}) = 0$$

空間反転

空間反転により、交換関係を不变にするため、

$$\mathcal{U}_p a^{(\lambda)}(\mathbf{p}) \mathcal{U}_p^{-1} = a^{(\lambda)}(-\mathbf{p}) = \eta_p a^{(\lambda)}(\mathbf{p}), \mathcal{U}_p a^{(\lambda)\dagger}(-\mathbf{p}) \mathcal{U}_p^{-1} = a^{(\lambda)\dagger}(-\mathbf{p}) = \eta_p^* a^{(\lambda)\dagger}(\mathbf{p}) \Leftarrow |\eta_p|^2 = 1$$

$e_\mu^{(1,2,3,4)}(\mathbf{p})$ はベクターなので

$$e_\mu^{(\lambda)}(-\mathbf{p}) \Rightarrow e_{1,2,3}^{(\lambda)}(-\mathbf{p}) = -e_{1,2,3}^{(\lambda)}(\mathbf{p}), e_0^{(\lambda)}(-\mathbf{p}) = e_0^{(\lambda)}(\mathbf{p}) \quad (\Leftarrow e_0^{(1,2,3)}(\mathbf{p}) = 0 \text{ となる})$$

従って

$$\begin{aligned} \mathcal{U}_p \mathbf{a}(\mathbf{p}) \mathcal{U}_p^{-1} &= \mathbf{a}(-\mathbf{p}) = \sum_{\lambda=1}^3 \mathbf{e}^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) = \sum_{\lambda=1}^3 (-\mathbf{e}^{(\lambda)}(\mathbf{p})) (\eta_p a^{(\lambda)}(\mathbf{p})) = -\eta_p \mathbf{a}(\mathbf{p}) \\ \mathcal{U}_p a_0(\mathbf{p}) \mathcal{U}_p^{-1} &= a_0(-\mathbf{p}) = \sum_{\lambda=1}^3 e_0^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) = \sum_{\lambda=1}^3 e_0^{(\lambda)}(\mathbf{p}) (\eta_p a^{(\lambda)}(\mathbf{p})) = \eta_p a_0(\mathbf{p}) \end{aligned}$$

空間反転は

$$\begin{aligned} \mathcal{U}_p A_0(x) \mathcal{U}_p^{-1} &\equiv A_0(t, -x) = \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} [a_0(-\mathbf{p}) \exp(-iEt + i(-\mathbf{p})(-x)) + a_0^\dagger(-\mathbf{p}) \exp(iEt - i(-\mathbf{p})(-x))] \\ &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [(\eta_p a_0(\mathbf{p})) \exp(-iEt + ipx) + (\eta_p a_0^\dagger(\mathbf{p})) \exp(iEt - ipx)] = \eta_p A_0(t, x) \quad (px = Et - px) \end{aligned}$$

$$\mathcal{U}_p \mathbf{A}(x) \mathcal{U}_p^{-1} \equiv \mathbf{A}(t, -x) = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\mathbf{a}(\mathbf{p}) \exp(-iEt + ip(-x)) + \mathbf{a}^\dagger(\mathbf{p}) \exp(iEt - ip(-x))]$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} [\mathbf{a}(-\mathbf{p}) \exp(-iEt + i(-\mathbf{p})(-\mathbf{x})) + \mathbf{a}^\dagger(\mathbf{p}) \exp(iEt - i(-\mathbf{p})(-\mathbf{x}))] \\
&= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [(-\eta_p \mathbf{a}(\mathbf{p})) \exp(-iEt + i\mathbf{p}\mathbf{x}) + (-\eta_p \mathbf{a}^\dagger(\mathbf{p})) \exp(iEt - i\mathbf{p}\mathbf{x})] = -\eta_p \mathbf{A}(t, \mathbf{x})
\end{aligned}$$

中性なので、 $\eta_p = \pm 1$ である：

$$\begin{aligned}
\mathbf{A}(t, \mathbf{x}) &\stackrel{\text{定義により}}{=} \mathcal{U}_P (\mathcal{U}_P \mathbf{A}(t, \mathbf{x}) \mathcal{U}_P^{-1}) \mathcal{U}_P^{-1} = \mathcal{U}_P (-\eta_p \mathbf{A}(t, \mathbf{x})) \mathcal{U}_P^{-1} = -\eta_p (-\eta_p \mathbf{A}(t, \mathbf{x})) \Rightarrow \eta_p = \pm 1 \\
A_0(t, \mathbf{x}) &\stackrel{\text{定義により}}{=} \mathcal{U}_P (\mathcal{U}_P A_0(t, \mathbf{x}) \mathcal{U}_P^{-1}) \mathcal{U}_P^{-1} = \mathcal{U}_P (\eta_p A_0(t, \mathbf{x})) \mathcal{U}_P^{-1} = \eta_p (\eta_p A_0(t, \mathbf{x})) \Rightarrow \eta_p = \pm 1
\end{aligned}$$

以上、まとめて、

$$\boxed{\mathcal{U}_P \mathbf{A}(x) \mathcal{U}_P^{-1} = \mathbf{A}(t, -\mathbf{x}) = -\eta_p \mathbf{A}(t, \mathbf{x}), \mathcal{U}_P A_0(x) \mathcal{U}_P^{-1} = A_0(t, -\mathbf{x}) = \eta_p A_0(t, \mathbf{x})}$$

$$\begin{aligned}
a^{(\lambda)}(-\mathbf{p}) &= \eta_p a^{(\lambda)}(\mathbf{p}) \Leftrightarrow |\eta_p|^2 = 1 \text{ for } \eta_p = +(\text{ベクトー}), \eta_p = -(\text{擬ベクトー}) \\
e_\mu^{(\lambda)}(-\mathbf{p}) &\Rightarrow e_{1,2,3}^{(\lambda)}(-\mathbf{p}) = -e_{1,2,3}^{(\lambda)}(\mathbf{p}), e_0^{(\lambda)}(-\mathbf{p}) = e_0^{(\lambda)}(\mathbf{p}) (\Leftrightarrow e_0^{(1,2,3)}(\mathbf{p}) = 0 \text{ にとれる})
\end{aligned}$$

(*) スピノの固有状態で表すと、 $\sigma \left(= \mathbf{p}^3 / |\mathbf{p}^3| \right) \rightarrow -\sigma$ より

$$\begin{aligned}
\mathcal{U}_P a^{(+)}(\mathbf{p}) \mathcal{U}_P^{-1} &= a^{(+)}(-\mathbf{p}) = \frac{a^{(1)}(-\mathbf{p}) + i(-\sigma) a^{(2)}(-\mathbf{p})}{\sqrt{2}} = \eta_p \frac{a^{(1)}(\mathbf{p}) - i\sigma a^{(2)}(\mathbf{p})}{\sqrt{2}} = \eta_p a^{(-)}(\mathbf{p}) \\
\mathcal{U}_P a^{(-)}(\mathbf{p}) \mathcal{U}_P^{-1} &= a^{(-)}(-\mathbf{p}) = \frac{a^{(1)}(-\mathbf{p}) - i(-\sigma) a^{(2)}(-\mathbf{p})}{\sqrt{2}} = \eta_p \frac{a^{(1)}(\mathbf{p}) + i\sigma a^{(2)}(\mathbf{p})}{\sqrt{2}} = \eta_p a^{(+)}(\mathbf{p}) \\
\mathcal{U}_P a^{(\parallel)}(\mathbf{p}) \mathcal{U}_P^{-1} &= a^{(\parallel)}(-\mathbf{p}) = (-\sigma) a^{(3)}(-\mathbf{p}) = -\sigma \eta_p a^{(3)}(\mathbf{p}) = -\eta_p a^{(\parallel)}(\mathbf{p}) \\
e^{(+)}(-\mathbf{p}) &= -\frac{e^{(1)}(\mathbf{p}) + i(-\sigma) e^{(2)}(\mathbf{p})}{\sqrt{2}} = -\frac{e^{(1)}(\mathbf{p}) - i\sigma e^{(2)}(\mathbf{p})}{\sqrt{2}} = -e^{(-)}(\mathbf{p}) \\
e^{(-)}(-\mathbf{p}) &= -\frac{e^{(1)}(\mathbf{p}) - i(-\sigma) e^{(2)}(\mathbf{p})}{\sqrt{2}} = -\frac{e^{(1)}(\mathbf{p}) + i\sigma e^{(2)}(\mathbf{p})}{\sqrt{2}} = -e^{(+)}(\mathbf{p}) \\
e^{(\parallel)}(-\mathbf{p}) &= (-\sigma) (-e^{(3)}(\mathbf{p})) = e^{(\parallel)}(\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}(\mathbf{p}) &= \sum_{\lambda=1}^3 e^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) = \frac{e^{(+)}(\mathbf{p}) + e^{(-)}(\mathbf{p})}{\sqrt{2}} a^{(1)}(\mathbf{p}) + \frac{e^{(+)}(\mathbf{p}) - e^{(-)}(\mathbf{p})}{\sqrt{2}i\sigma} a^{(2)}(\mathbf{p}) + \sigma e^{(\parallel)}(\mathbf{p}) a^{(3)}(\mathbf{p}) \\
&= e^{(-)}(\mathbf{p}) \frac{a^{(1)}(\mathbf{p}) + i\sigma a^{(2)}(\mathbf{p})}{\sqrt{2}} + e^{(+)}(\mathbf{p}) \frac{a^{(1)}(\mathbf{p}) - i\sigma a^{(2)}(\mathbf{p})}{\sqrt{2}} + e^{(\parallel)}(\mathbf{p}) \sigma a^{(3)}(\mathbf{p}) \\
&= e^{(-)}(\mathbf{p}) a^{(+)}(\mathbf{p}) + e^{(+)}(\mathbf{p}) a^{(-)}(\mathbf{p}) + e^{(\parallel)}(\mathbf{p}) a^{(\parallel)}(\mathbf{p}) \quad \Rightarrow \text{物理自由度} \quad e^{(-)}(\mathbf{p}) a^{(+)}(\mathbf{p}) + e^{(+)}(\mathbf{p}) a^{(-)}(\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}(-\mathbf{p}) &= e^{(-)}(-\mathbf{p}) a^{(+)}(-\mathbf{p}) + e^{(+)}(-\mathbf{p}) a^{(-)}(-\mathbf{p}) + e^{(\parallel)}(-\mathbf{p}) a^{(\parallel)}(-\mathbf{p}) \\
&= (-e^{(+)}(\mathbf{p})) (\eta_p a^{(-)}(\mathbf{p})) + (-e^{(-)}(\mathbf{p})) (\eta_p a^{(+)}(\mathbf{p})) + (e^{(\parallel)}(\mathbf{p})) (-\eta_p a^{(\parallel)}(\mathbf{p})) \\
&= -\eta_p (e^{(+)}(\mathbf{p}) a^{(-)}(\mathbf{p}) + e^{(-)}(\mathbf{p}) a^{(+)}(\mathbf{p}) + e^{(\parallel)}(\mathbf{p}) a^{(\parallel)}(\mathbf{p})) = -\eta_p \mathbf{a}(\mathbf{p})
\end{aligned}$$

スピノ 1/2

4つのスピノール $u_{\alpha}^{(1,2,3,4)}(\mathbf{p})$ はディラック方程式を満たす：

$$u_{\alpha}^{(1)}(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, u_{\alpha}^{(2)}(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, u_{\alpha}^{(3)}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_{\alpha}^{(4)}(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } \begin{cases} \alpha = 1 \\ \alpha = 2 \\ \alpha = 3 \\ \alpha = 4 \end{cases}$$

を用いると、対応する4つの生成・消滅演算子 $a^{(1,2,3,4)}(\mathbf{p}), a^{(1,2,3,4)\dagger}(\mathbf{p})$ を用いて

$$a_{\alpha}(\mathbf{p}) = \sum_{\lambda=1}^4 u_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) = \begin{pmatrix} a^{(1)}(\mathbf{p}) \\ a^{(2)}(\mathbf{p}) \\ a^{(3)}(\mathbf{p}) \\ a^{(4)}(\mathbf{p}) \end{pmatrix}, \quad \{a^{(\lambda)}(\mathbf{p}), a^{(\lambda')\dagger}(\mathbf{p})\} = \delta_{\lambda\lambda'}$$

とする。交換関係は、

$$\{a_{\alpha}(\mathbf{p}), a_{\beta}^{\dagger}(\mathbf{p})\} = \left\{ \sum_{\lambda=1}^4 e_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}), \sum_{\lambda'=1}^4 e_{\beta}^{(\lambda')*}(\mathbf{p}) a^{(\lambda')\dagger}(\mathbf{p}) \right\} = \sum_{\lambda=1}^4 \delta_{\lambda\lambda'} e_{\alpha}^{(\lambda)}(\mathbf{p}) e_{\beta}^{(\lambda)*}(\mathbf{p}) = \delta_{\alpha\beta}$$

空間反転の性質を見るため、自由粒子のディラック方程式を解く。そのため、

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}^i \\ -\boldsymbol{\sigma}^i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} (\mathbf{p}=0 \text{ で粒子が上2成分に収まる表示})$$

とする。詳しくは、Appendix 1 を参照。

反粒子

さて、Appendix 2 で求めてある解 $\varphi_{\alpha}^{(\lambda)}$ を使って、

$$\psi_{\alpha}(x) \sim \int \frac{d\mathbf{p}}{2E} a_{\alpha}(\mathbf{p}) \exp(-ipx) = \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^4 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx)$$

を考えると、 $\varphi_{\alpha}^{(\lambda=3,4)}$ がエネルギー負の解であるので、

$$\begin{aligned} \psi_{\alpha}(x) &\sim \int d\mathbf{p} \delta(p^0 - |E|) \sum_{\lambda=1}^4 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) \\ &= \int d\mathbf{p} \frac{\delta(p^0 - |E|)}{2p^0} \sum_{\lambda=1}^2 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) + \int d\mathbf{p} \frac{\delta(p^0 + |E|)}{2p^0} \sum_{\lambda=3}^4 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) \\ &= \int \frac{d\mathbf{p}}{2|E|} \sum_{\lambda=1}^2 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-i|E|t + ipx) + \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{-2|E|} \sum_{\lambda=3}^4 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-i(|E|t + ipx)) \\ &= \int \frac{d\mathbf{p}}{2|E|} \sum_{\lambda=1}^2 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-i|E|t + ipx) + \int_{-\infty}^{-\infty} \frac{d(-\mathbf{p})}{-2|E|} \sum_{\lambda=3}^4 \varphi_{\alpha}^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) \exp(i|E|t + i(-\mathbf{p})x) \\ &= \int \frac{d\mathbf{p}}{2|E|} \sum_{\lambda=1}^2 \varphi_{\alpha}^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-i|E|t + ipx) + \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2|E|} \sum_{\lambda=3}^4 \varphi_{\alpha}^{(\lambda)}(-\mathbf{p}) (-a^{(\lambda)}(-\mathbf{p})) \exp(i|E|t - ipx) \end{aligned}$$

ここで、反粒子の生成演算子として

$$b^{(\lambda)\dagger}(\mathbf{p}) = -a^{(\lambda)}(-\mathbf{p}) \Leftrightarrow \left\{ a^{(\lambda)}(-\mathbf{p}), a^{(\lambda')\dagger}(-\mathbf{p}) \right\} = \left\{ b^{(\lambda)\dagger}(\mathbf{p}), b^{(\lambda')\dagger}(\mathbf{p}) \right\} = \delta^{\lambda\lambda'}$$

とするとき、 $E > 0$ として

$$\begin{aligned} \psi_\alpha(x) &\sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-i|E|t + i\mathbf{p}\mathbf{x}) + \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(i|E|t - i\mathbf{p}\mathbf{x}) \right] \\ &= \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) + \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) \right] \end{aligned}$$

なので、

$$\boxed{\psi_\alpha(x) \sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) + \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) \right]}$$

従って、あらたに、粒子・反粒子の生成・消滅演算子として

$$a_\alpha(\mathbf{p}) = \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}), \quad b_\alpha^\dagger(\mathbf{p}) = \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p})$$

とすると、

$$\boxed{\psi_\alpha(x) \sim \int \frac{d\mathbf{p}}{2E} [a_\alpha(\mathbf{p}) \exp(-ipx) + b_\alpha^\dagger(\mathbf{p}) \exp(ipx)]}$$

と表記される。(以降、添え字 α は必要なとき以外は落とす)

空間反転

空間反転 ($\mathbf{p} \rightarrow -\mathbf{p}$) による $\varphi_\alpha^{(\lambda=1,2,3,4)}$ の解の振る舞いは (\mathbf{x} は規格化因子で運動量に依存) · · ·

$$\begin{aligned} \varphi^{(2) \text{ with } E>0, S=-1}(-\mathbf{p}) &= \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \end{pmatrix}_x \Bigg|_{\mathbf{p} \rightarrow -\mathbf{p}} = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \end{pmatrix}_x \\ &= \begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & -\mathbf{I} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \end{pmatrix} \Bigg|_{\mathbf{p} \rightarrow -\mathbf{p}} = \gamma^0 \varphi^{(1) \text{ with } E>0, S=1}(\mathbf{p}) \end{aligned}$$

$$\varphi^{(2)}(-\mathbf{p}) = \gamma^0 \varphi^{(1)}(\mathbf{p}), \quad \varphi^{(1)}(-\mathbf{p}) = \gamma^0 \varphi^{(2)}(\mathbf{p})$$

同様に · · ·

$$\begin{aligned}
\varphi^{(4) \text{ with } E < 0, S = -1}(-\mathbf{p}) &= \begin{pmatrix} \frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \\ 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix}_{\mathbf{p} \rightarrow -\mathbf{p}} x = \begin{pmatrix} \frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x \\
&= -\begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathbf{I} \end{pmatrix} \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x = -\gamma^0 \varphi^{(3) \text{ with } E < 0, S = 1}(\mathbf{p})
\end{aligned}$$

$$\varphi^{(4)}(-\mathbf{p}) = -\gamma^0 \varphi^{(3)}(\mathbf{p}), \quad \varphi^{(3)}(-\mathbf{p}) = -\gamma^0 \varphi^{(4)}(\mathbf{p})$$

を得る。また、交換関係を不変にするために、上述の解の振る舞い

$$\boxed{\varphi^{(1)}(-\mathbf{p}) = \gamma^0 \varphi^{(2)}(\mathbf{p}), \quad \varphi^{(2)}(-\mathbf{p}) = \gamma^0 \varphi^{(1)}(\mathbf{p}), \quad \varphi^{(3)}(-\mathbf{p}) = -\gamma^0 \varphi^{(4)}(\mathbf{p}), \quad \varphi^{(4)}(-\mathbf{p}) = -\gamma^0 \varphi^{(3)}(\mathbf{p})}$$

も考慮し

$$\begin{aligned}
\mathcal{U}_P a^{(1)}(\mathbf{p}) \mathcal{U}_P^{-1} &= a^{(1)}(-\mathbf{p}) = \eta_p a^{(2)}(\mathbf{p}), \quad \mathcal{U}_P a^{(2)}(\mathbf{p}) \mathcal{U}_P^{-1} = a^{(2)}(-\mathbf{p}) = \eta_p a^{(1)}(\mathbf{p}) \\
\mathcal{U}_P b^{(1)\dagger}(\mathbf{p}) \mathcal{U}_P^{-1} &= b^{(1)\dagger}(-\mathbf{p}) = \eta'_p b^{(2)\dagger}(\mathbf{p}), \quad \mathcal{U}_P b^{(2)\dagger}(\mathbf{p}) \mathcal{U}_P^{-1} = b^{(2)\dagger}(-\mathbf{p}) = \eta'_p b^{(1)\dagger}(\mathbf{p})
\end{aligned}$$

とする。

以上から、

$$\begin{aligned}
a_\alpha(\mathbf{p}) &= \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \Rightarrow \mathcal{U}_P a_\alpha(\mathbf{p}) \mathcal{U}_P^{-1} = a_\alpha(-\mathbf{p}) = \varphi_\alpha^{(1)}(-\mathbf{p}) a^{(1)}(-\mathbf{p}) + \varphi_\alpha^{(2)}(-\mathbf{p}) a^{(2)}(-\mathbf{p}) \\
&= (\gamma^0)_{\alpha\beta} \varphi_\beta^{(2)}(\mathbf{p}) (\eta_p a^{(2)}(\mathbf{p})) + (\gamma^0)_{\alpha\beta} \varphi_\beta^{(1)}(\mathbf{p}) (\eta_p a^{(1)}(\mathbf{p})) = \eta_p (\gamma^0)_{\alpha\beta} a_\beta(\mathbf{p}) \\
b_\alpha^\dagger(\mathbf{p}) &= \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \Rightarrow \mathcal{U}_P b_\alpha^\dagger(\mathbf{p}) \mathcal{U}_P^{-1} = b_\alpha^\dagger(-\mathbf{p}) = \varphi_\alpha^{(3)}(\mathbf{p}) b^{(1)\dagger}(-\mathbf{p}) + \varphi_\alpha^{(4)}(\mathbf{p}) b^{(2)\dagger}(-\mathbf{p}) \\
&= -(\gamma^0)_{\alpha\beta} \varphi_\beta^{(4)}(\mathbf{p}) (\eta_p b^{(2)\dagger}(\mathbf{p})) - (\gamma^0)_{\alpha\beta} \varphi_\beta^{(3)}(\mathbf{p}) (\eta_p b^{(1)\dagger}(\mathbf{p})) = -\eta'_p (\gamma^0)_{\alpha\beta} \gamma^0 b_\beta^\dagger(\mathbf{p})
\end{aligned}$$

従って、空間反転は、

$$\begin{aligned}
\mathcal{U}_P \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} &\equiv \psi(t, -\mathbf{x}) \sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-i|E|t + i\mathbf{p}(-\mathbf{x})) + b^\dagger(\mathbf{p}) \exp(i|E|t - i\mathbf{p}(-\mathbf{x}))] \\
&= \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} [a(-\mathbf{p}) \exp(-i|E|t + i(-\mathbf{p})(-\mathbf{x})) + b^\dagger(-\mathbf{p}) \exp(i|E|t - i(-\mathbf{p})(-\mathbf{x}))] \\
&= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(-\mathbf{p}) \exp(-i|E|t + i\mathbf{p}\mathbf{x}) + b^\dagger(-\mathbf{p}) \exp(i|E|t - i\mathbf{p}\mathbf{x})]
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_p \gamma^0 a(\mathbf{p}) \exp(-i|E|t + i\mathbf{p}\mathbf{x}) - \eta'_p \gamma^0 b^\dagger(\mathbf{p}) \exp(i|E|t - i\mathbf{p}\mathbf{x})]$$

これが元の場で書けるには、 $\eta'_p = -\eta_p$ の必要がある。その結果、

$$\begin{aligned} \psi(t, -\mathbf{x}) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_p \gamma^0 a(\mathbf{p}) \exp(-i|E|t + i\mathbf{p}\mathbf{x}) - \eta'_p \gamma^0 b^\dagger(\mathbf{p}) \exp(i|E|t - i\mathbf{p}\mathbf{x})] \\ &= \eta_p \gamma^0 \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-i|E|t + i\mathbf{p}\mathbf{x}) + b^\dagger(\mathbf{p}) \exp(i|E|t - i\mathbf{p}\mathbf{x})] = \eta_p \gamma^0 \psi(t, \mathbf{x}) \\ \overline{\psi_\alpha(t, -\mathbf{x})} &= \psi_\alpha^\dagger(t, -\mathbf{x}) \gamma^0 = \eta_p^* \psi_\alpha^\dagger(t, \mathbf{x}) \gamma^0 \gamma^0 = \eta_p^* \psi_\alpha^\dagger(t, \mathbf{x}) \end{aligned}$$

また、 $\eta'_p = -\eta_p$ より

$$a_\alpha(-\mathbf{p}) = \eta_p \gamma^0 a_\alpha(\mathbf{p}), b_\alpha^\dagger(-\mathbf{p}) = -\eta_p \gamma^0 b_\alpha^\dagger(\mathbf{p})$$

以上から、まとめると、

$$\boxed{\begin{aligned} \mathcal{U}_P \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} &= \psi(t, -\mathbf{x}) = \eta_p \gamma^0 \psi(t, \mathbf{x}), \quad \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \mathcal{U}_P^{-1} = \overline{\psi(t, -\mathbf{x})} = \eta_p^* \psi^\dagger(t, \mathbf{x}) \\ \mathcal{U}_P a(\mathbf{p}) \mathcal{U}_P^{-1} &= a(-\mathbf{p}) = \eta_p \gamma^0 a(\mathbf{p}), \quad \mathcal{U}_P b^\dagger(\mathbf{p}) \mathcal{U}_P^{-1} = b^\dagger(-\mathbf{p}) = -\eta_p \gamma^0 b^\dagger(\mathbf{p}) \\ a_\alpha(\mathbf{p}) &= \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}), \quad b_\alpha^\dagger(\mathbf{p}) = \sum_{\lambda=1}^2 \varphi_\alpha^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \\ a^{(1)}(-\mathbf{p}) &= \eta_p a^{(2)}(\mathbf{p}), \quad a^{(2)}(-\mathbf{p}) = \eta_p a^{(1)}(\mathbf{p}), \quad b^{(1)\dagger}(-\mathbf{p}) = \eta_p b^{(2)\dagger}(\mathbf{p}), \quad b^{(2)\dagger}(-\mathbf{p}) = \eta_p b^{(1)\dagger}(\mathbf{p}) \\ \varphi^{(2)}(-\mathbf{p}) &= \gamma^0 \varphi^{(1)}(\mathbf{p}), \quad \varphi^{(1)}(-\mathbf{p}) = \gamma^0 \varphi^{(2)}(\mathbf{p}), \quad \varphi^{(4)}(-\mathbf{p}) = -\gamma^0 \varphi^{(3)}(\mathbf{p}), \quad \varphi^{(3)}(-\mathbf{p}) = -\gamma^0 \varphi^{(4)}(\mathbf{p}) \end{aligned}}$$

第二節：時間反転（T変換） $t \rightarrow -t$

$\mathbf{p} \rightarrow -\mathbf{p}$ (運動量), $\mathbf{L} \rightarrow -\mathbf{L}$ (角運動量 $\mathbf{x} \times \mathbf{p}$), $\mathbf{S} \rightarrow -\mathbf{S}$ (スピン), $h \rightarrow h$ (ヘリシティ)

量子力学において

シュレディンガー方程式は

$$i \frac{\partial \psi(t, \mathbf{x})}{\partial t} = H(t) \psi(t, \mathbf{x})$$

で、 $t \rightarrow -t$ にすると、

$$i \frac{\partial \psi(-t, \mathbf{x})}{\partial(-t)} = H(-t) \psi(-t, \mathbf{x}) \Rightarrow -i \frac{\partial \psi(-t, \mathbf{x})}{\partial t} = H(-t) \psi(-t, \mathbf{x})$$

一方、複素共役をとると、

$$\begin{aligned} \left(i \frac{\partial \psi(t, \mathbf{x})}{\partial t} \right)^\dagger &= (H(t) \psi(t, \mathbf{x}))^\dagger \Rightarrow -i \frac{\partial \psi^\dagger(t, \mathbf{x})}{\partial t} = \psi^\dagger(t, \mathbf{x}) H^\dagger(t) \stackrel{H=H^\dagger}{=} \psi^\dagger(t, \mathbf{x}) H(t) \\ \xrightarrow{\text{Transpose}} -i \left(\frac{\partial \psi^\dagger(t, \mathbf{x})}{\partial t} \right)^T &= (\psi^\dagger(t, \mathbf{x}) H(t))^T \Rightarrow -i \frac{\partial \psi^*(t, \mathbf{x})}{\partial t} = H^T(t) \psi^*(t, \mathbf{x}) \end{aligned}$$

$$\xrightarrow{\text{Unitary演算子}} -i \frac{\partial U_T \psi^*(t, \mathbf{x})}{\partial t} = U_T H^T(t) U_T^{-1} U_T \psi^*(t, \mathbf{x}) \Leftarrow \text{任意性の導入}$$

になる。そこで、

$$-i \frac{\partial \psi(-t, \mathbf{x})}{\partial t} = H(-t) \psi(-t, \mathbf{x}) \Leftrightarrow -i \frac{\partial U_T \psi^*(t, \mathbf{x})}{\partial t} = U_T H^T(t) U_T^{-1} U_T \psi^*(t, \mathbf{x})$$

より、時間反転の演算子を \mathcal{U}_T とすると

$$\mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} \equiv \psi(-t, \mathbf{x}) = U_T \psi^*(t, \mathbf{x}) \quad \text{with } \mathcal{U}_T H(t) \mathcal{U}_T^{-1} \equiv H(-t) = U_T H^T(t) U_T^{-1}$$

の対応ができる。再度、 $t \rightarrow -t$ をすると、位相を除いて元に戻るので、位相を η とし

$$\begin{aligned} \eta \psi(x) &\equiv \mathcal{U}_T \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} \mathcal{U}_T^{-1} = \mathcal{U}_T U_T \psi^*(x) \mathcal{U}_T^{-1} = U_T \mathcal{U}_T \psi^*(x) \mathcal{U}_T^{-1} (= U_T \psi^*(-t, \mathbf{x})) \\ &= U_T \mathcal{U}_T (U_T \psi^*(x))^* \mathcal{U}_T^{-1} = U_T U_T^* \psi(x) \Rightarrow U_T U_T^* = \eta \quad (\text{with } |\eta|^2 = 1) \\ \mathcal{U}_T \psi^*(x) \mathcal{U}_T^{-1} &\equiv \psi^*(-t, \mathbf{x}) = \eta U_T^{-1} \psi(x) \end{aligned}$$

ここで、この位相分の自由度が無矛盾なためには

$$\begin{aligned} \psi(-t, \mathbf{x}) &= U_T \psi^*(t, \mathbf{x}) \Rightarrow \psi^*(-t, \mathbf{x}) = U_T^* \psi(t, \mathbf{x}) \xrightarrow{U_T \text{を乗ずる}} U_T \psi^*(-t, \mathbf{x}) = U_T U_T^* \psi(t, \mathbf{x}) = \eta \psi(t, \mathbf{x}) \\ &\Rightarrow \psi(t, \mathbf{x}) = \eta^{-1} U_T \psi^*(-t, \mathbf{x}) \text{ vs } U_T^{*-1} \psi^*(-t, \mathbf{x}) = \psi(t, \mathbf{x}) \Rightarrow U_T U_T^* = \eta \Rightarrow \text{OK} \\ \eta \psi(t, \mathbf{x}) &= U_T \psi^*(-t, \mathbf{x}) \Rightarrow \eta^* \psi^*(t, \mathbf{x}) = U_T^* \psi(-t, \mathbf{x}) \xrightarrow{U_T \text{を乗ずる}} \eta^* U_T \psi^*(t, \mathbf{x}) = U_T U_T^* \psi(-t, \mathbf{x}) = \eta \psi(-t, \mathbf{x}) \\ &\Rightarrow \psi(-t, \mathbf{x}) = \eta^{-1} \eta^* U_T \psi^*(t, \mathbf{x}) \text{ vs } \psi(-t, \mathbf{x}) = U_T \psi^*(t, \mathbf{x}) \Rightarrow \eta = \eta^* \Rightarrow [\eta = \pm 1] \end{aligned}$$

以上から、

$$\boxed{\begin{aligned} \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} &\equiv \psi(-t, \mathbf{x}) = U_T \psi^*(t, \mathbf{x}) \quad (*) \eta = \pm 1 \\ \mathcal{U}_T \psi^*(x) \mathcal{U}_T^{-1} &\equiv \psi^*(-t, \mathbf{x}) = \eta U_T^{-1} \psi(x) \quad (\eta \psi(x) = U_T \psi^*(-t, \mathbf{x})) \end{aligned} \quad \text{with } \mathcal{U}_T H(t) \mathcal{U}_T^{-1} \equiv H(-t) = U_T H^T(t) U_T^{-1}}$$

がわかる。この運動量表示では、

$$\begin{aligned} \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} &\equiv \psi(-t, \mathbf{x}) = U_T \psi^*(t, \mathbf{x}) = U_T \int_{-\infty}^{\infty} d\mathbf{p} \varphi^*(\mathbf{p}) \exp(ipx) \quad (px = Et - p\mathbf{x}) \\ \psi(-t, \mathbf{x}) &= \int d\mathbf{p} \varphi(\mathbf{p}) \exp(-iE(-t) + ip\mathbf{x}) = \int d\mathbf{p} \varphi(\mathbf{p}) \exp(iEt + ip\mathbf{x}) \\ &= \int_{-\infty}^{-\infty} d(-\mathbf{p}) \varphi(-\mathbf{p}) \exp(iEt + i(-\mathbf{p})\mathbf{x}) = \int_{-\infty}^{\infty} d\mathbf{p} \varphi(-\mathbf{p}) \exp(iEt - ip\mathbf{x}) \\ \Rightarrow \mathcal{U}_T \varphi(\mathbf{p}) \mathcal{U}_T^{-1} &\equiv \varphi(-\mathbf{p}) = U_T \varphi^*(\mathbf{p}) \end{aligned}$$

従って、

$$\boxed{\mathcal{U}_T \varphi(\mathbf{p}) \mathcal{U}_T^{-1} \equiv \varphi(-\mathbf{p}) = U_T \varphi^*(\mathbf{p})}$$

スピノ

$\varphi(-\mathbf{p}) = U_T \varphi^*(\mathbf{p})$ に従って、粒子・反粒子の消滅・生成演算子に対して

粒子 : $\mathcal{U}_T a(\mathbf{p}) \mathcal{U}_T^{-1} \equiv a(-\mathbf{p}) = \eta_T a^\dagger(\mathbf{p})$, 反粒子 : $\mathcal{U}_T b(\mathbf{p}) \mathcal{U}_T^{-1} \equiv b(-\mathbf{p}) = \eta_T^* b^\dagger(\mathbf{p}) \Leftarrow |\eta_T|^2 = 1$

と設定する。スピン 0 ので、内部自由度は無い。従って、 $U=1$ 。量子場は、

$$\begin{aligned}
 \phi(x) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-ipx) + b^\dagger(\mathbf{p}) \exp(ipx)] = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-iEt + ipx) + b^\dagger(\mathbf{p}) \exp(iEt - ipx)] \\
 \mathcal{U}_T \phi(x) \mathcal{U}_T^{-1} &\equiv \phi(-t, \mathbf{x}) \sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-iE(-t) + ipx) + b^\dagger(\mathbf{p}) \exp(iE(-t) - ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(iEt + ipx) + b^\dagger(\mathbf{p}) \exp(-iEt - ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(-\mathbf{p}) \exp(iEt + i(-\mathbf{p})\mathbf{x}) + b^\dagger(-\mathbf{p}) \exp(-iEt - i(-\mathbf{p})\mathbf{x})] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a(-\mathbf{p}) \exp(iEt - ipx) + b^\dagger(-\mathbf{p}) \exp(-iEt + ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_T a^\dagger(\mathbf{p}) \exp(iEt - ipx) + \eta_T b(\mathbf{p}) \exp(-iEt + ipx)] \\
 &= \eta_T \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a^\dagger(\mathbf{p}) \exp(iEt - ipx) + b(\mathbf{p}) \exp(-iEt + ipx)] = \eta_T \phi^\dagger(t, \mathbf{x}) \\
 \boxed{\mathcal{U}_T \phi(x) \mathcal{U}_T^{-1} \equiv \phi(-t, \mathbf{x}) = \eta_T \phi^\dagger(t, \mathbf{x}), \mathcal{U}_T \phi^\dagger(x) \mathcal{U}_T^{-1} \equiv \phi^\dagger(-t, \mathbf{x}) = \eta_T^* \phi(t, \mathbf{x})}
 \end{aligned}$$

スピン 1 (光子)

$\varphi(-\mathbf{p}) = U\varphi^*(\mathbf{p})$ に従って、粒子の消滅・生成演算子に対して

$$\begin{aligned}
 \text{粒子 : } \mathcal{U}_T a^{(\lambda)}(\mathbf{p}) \mathcal{U}_T^{-1} &\equiv a^{(\lambda)}(-\mathbf{p}) = \eta_T a^{(\lambda)\dagger}(\mathbf{p}), \mathcal{U}_T a^{(\lambda)\dagger}(\mathbf{p}) \mathcal{U}_T^{-1} \equiv a^{(\lambda)\dagger}(-\mathbf{p}) = \eta_T^* a^{(\lambda)}(\mathbf{p}) \Leftrightarrow |\eta_T|^2 = 1 \\
 e^{(\lambda)}(-\mathbf{p}) &= -e^{(\lambda)}(\mathbf{p}) = -e^{(\lambda)*}(\mathbf{p}), e_0^{(\lambda)}(-\mathbf{p}) = e_0^{(\lambda)}(\mathbf{p}) = e_0^{(\lambda)*}(\mathbf{p})
 \end{aligned}$$

を用いると、

$$\begin{aligned}
 \mathbf{A}(x) &\sim \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\mathbf{a}(\mathbf{p}) \exp(-ipx) + \mathbf{a}^\dagger(\mathbf{p}) \exp(ipx)] \quad (px = Et - p\mathbf{x}) \\
 \mathcal{U}_T \mathbf{A}(t, \mathbf{x}) \mathcal{U}_T^{-1} &\equiv \mathbf{A}(-t, \mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\mathbf{a}(\mathbf{p}) \exp(-iE(-t) + ipx) + \mathbf{a}^\dagger(\mathbf{p}) \exp(iE(-t) - ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} [\mathbf{a}(-\mathbf{p}) \exp(iEt + i(-\mathbf{p})\mathbf{x}) + \mathbf{a}^\dagger(-\mathbf{p}) \exp(-iEt - i(-\mathbf{p})\mathbf{x})] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\mathbf{a}(-\mathbf{p}) \exp(iEt - ipx) + \mathbf{a}^\dagger(-\mathbf{p}) \exp(-iEt + ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [e^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) \exp(ipx) + e^{(\lambda)*}(-\mathbf{p}) a^{(\lambda)\dagger}(-\mathbf{p}) \exp(-ipx)] \\
 &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [-e^{(\lambda)*}(\mathbf{p}) \eta_T a^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) - e^{(\lambda)}(\mathbf{p}) \eta_T^* a^{(\lambda)}(\mathbf{p}) \exp(-ipx)] \\
 &= -\int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\eta_T e^{(\lambda)*}(\mathbf{p}) a^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) + \eta_T^* e^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx)]
 \end{aligned}$$

これは、 $\eta_T = \eta_T^*$ のときに、まとめられる：

$$\mathcal{U}_T \mathbf{A}(t, \mathbf{x}) \mathcal{U}_T^{-1} \equiv \mathbf{A}(-t, \mathbf{x}) = -\eta_T \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [\mathbf{a}^\dagger(\mathbf{p}) \exp(ipx) + \mathbf{a}(\mathbf{p}) \exp(-ipx)] = -\eta_T \mathbf{A}(t, \mathbf{x})$$

同様に、

$$\begin{aligned}\mathcal{U}_T A_0(t, \mathbf{x}) \mathcal{U}_T^{-1} &\equiv A_0(-t, \mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a_0(\mathbf{p}) \exp(-iE(-t) + i\mathbf{p}\mathbf{x}) + a_0^\dagger(\mathbf{p}) \exp(iE(-t) - i\mathbf{p}\mathbf{x})] \\ &= \eta_T \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a_0^\dagger(\mathbf{p}) \exp(ipx) + a_0(\mathbf{p}) \exp(-ipx)] = \eta_T A_0(t, \mathbf{x})\end{aligned}$$

を得る。中性なので、空間反転と同様に $\eta_T = 1$ である。従って、

$$\begin{aligned}&[\mathcal{U}_T A(t, \mathbf{x}) \mathcal{U}_T^{-1} \equiv A(-t, \mathbf{x}) = -\eta_T A(t, \mathbf{x}), \mathcal{U}_T A_0(t, \mathbf{x}) \mathcal{U}_T^{-1} \equiv A_0(-t, \mathbf{x}) = \eta_T A_0(t, \mathbf{x})] \quad (\eta_T = \pm 1) \\ &\mathcal{U}_T a^{(\lambda)}(-\mathbf{p}) \mathcal{U}_T^{-1} \equiv a^{(\lambda)}(-\mathbf{p}) = \eta_T a^{(\lambda)\dagger}(\mathbf{p}) \\ &a_{1,2,3}^{(\lambda)}(-\mathbf{p}) = -e_{1,2,3}^{(\lambda)}(\mathbf{p}) = -e_{1,2,3}^{(\lambda)*}(\mathbf{p}), e_0^{(\lambda)}(-\mathbf{p}) = e_0^{(\lambda)}(\mathbf{p}) = e_0^{(\lambda)*}(\mathbf{p}) \quad (\Leftarrow e_0^{(1,2,3)}(\mathbf{p}) = 0 \text{ にとれる})\end{aligned}$$

である。スピンの固有状態での対応する変換は

$$\begin{aligned}\mathbf{e}^{(\pm)}(\mathbf{p}) &= \frac{\mathbf{e}^{(1)}(\mathbf{p}) \pm i\sigma(\mathbf{p})\mathbf{e}^{(2)}(\mathbf{p})}{\sqrt{2}}, \quad \mathbf{e}^{(\not\parallel)}(\mathbf{p}) = \sigma(\mathbf{p})\mathbf{e}^{(3)}(\mathbf{p}) \quad \left(\sigma(\mathbf{p}) = \frac{\mathbf{p}^3}{|\mathbf{p}^3|} = \pm 1 \right) \\ \mathbf{a}(\mathbf{p}) &= \mathbf{e}^{(-)}(\mathbf{p})\mathbf{a}^{(+)}(\mathbf{p}) + \mathbf{e}^{(+)}(\mathbf{p})\mathbf{a}^{(-)}(\mathbf{p}) + \mathbf{e}^{(\not\parallel)}(\mathbf{p})\mathbf{a}^{(\not\parallel)}(\mathbf{p}) \\ \mathbf{a}^{(+)}(-\mathbf{p}) &= \frac{\mathbf{a}^{(1)}(-\mathbf{p}) + i(-\sigma)\mathbf{a}^{(2)}(-\mathbf{p})}{\sqrt{2}} = \eta_T \frac{\mathbf{a}^{(1)\dagger}(\mathbf{p}) - i\sigma\mathbf{a}^{(2)\dagger}(\mathbf{p})}{\sqrt{2}} = \eta_T \left(\frac{\mathbf{a}^{(1)}(\mathbf{p}) + i\sigma\mathbf{a}^{(2)}(\mathbf{p})}{\sqrt{2}} \right)^\dagger = \eta_T \mathbf{a}^{(+)\dagger}(\mathbf{p}) \\ \mathbf{a}^{(-)}(-\mathbf{p}) &= \frac{\mathbf{a}^{(1)}(-\mathbf{p}) - i(-\sigma)\mathbf{a}^{(2)}(-\mathbf{p})}{\sqrt{2}} = \eta_p \frac{\mathbf{a}^{(1)\dagger}(\mathbf{p}) + i\sigma\mathbf{a}^{(2)\dagger}(\mathbf{p})}{\sqrt{2}} = \eta_T \left(\frac{\mathbf{a}^{(1)}(\mathbf{p}) - i\sigma\mathbf{a}^{(2)}(\mathbf{p})}{\sqrt{2}} \right)^\dagger = \eta_T \mathbf{a}^{(-)\dagger}(\mathbf{p}) \\ \mathbf{a}^{(\not\parallel)}(-\mathbf{p}) &= (-\sigma)\mathbf{a}^{(3)}(-\mathbf{p}) = -\sigma\eta_T \mathbf{a}^{(3)\dagger}(\mathbf{p}) = -\eta_T \mathbf{a}^{(\not\parallel)\dagger}(\mathbf{p}) \\ \mathbf{e}^{(\pm)}(-\mathbf{p}) &= \frac{\mathbf{e}^{(1)}(-\mathbf{p}) \pm i\sigma(-\mathbf{p})\mathbf{e}^{(2)}(-\mathbf{p})}{\sqrt{2}} = \frac{(-\mathbf{e}^{(1)}(\mathbf{p})) \pm i(-\sigma(\mathbf{p}))(-\mathbf{e}^{(2)}(\mathbf{p}))}{\sqrt{2}} = \frac{-\mathbf{e}^{(1)}(\mathbf{p}) \pm i\sigma(\mathbf{p})\mathbf{e}^{(2)}(\mathbf{p})}{\sqrt{2}} = \\ &= -\frac{\mathbf{e}^{(1)}(\mathbf{p}) \mp i\sigma(\mathbf{p})\mathbf{e}^{(2)}(\mathbf{p})}{\sqrt{2}} = -\left(\frac{\mathbf{e}^{(1)}(\mathbf{p}) \pm i\sigma(\mathbf{p})\mathbf{e}^{(2)}(\mathbf{p})}{\sqrt{2}} \right)^* = -\mathbf{e}^{(\pm)*}(\mathbf{p}) \\ \mathbf{e}^{(\not\parallel)}(-\mathbf{p}) &= \sigma(-\mathbf{p})\mathbf{e}^{(3)}(-\mathbf{p}) = (-\sigma(\mathbf{p}))(-\mathbf{e}^{(3)}(\mathbf{p})) = \mathbf{e}^{(\not\parallel)}(\mathbf{p}) = \mathbf{e}^{(\not\parallel)*}(\mathbf{p})\end{aligned}$$

になる。

スピン 1/2

ディラック方程式での時間反転

ディラック方程式での時間反転の影響を調べると・・・

$$(i\partial_\mu \gamma^\mu - m)_{\alpha\beta} \psi_\beta(t, \mathbf{x}) = \left(i\gamma^0 \frac{\partial}{\partial t} - i\gamma^i \nabla^i - m \right)_{\alpha\beta} \psi_\beta(t, \mathbf{x}) = 0$$

で、 $t \rightarrow -t$ にすると、

$$\left(i\gamma^0 \frac{\partial}{\partial(-t)} - i\gamma^i \nabla^i - m \right)_{\alpha\beta} \psi_\beta(-t, \mathbf{x}) = 0 \Rightarrow \left(-i\gamma^0 \frac{\partial}{\partial t} - i\gamma^i \nabla^i - m \right)_{\alpha\beta} \psi_\beta(-t, \mathbf{x}) = 0$$

一方、複素共役をとると、

$$\begin{aligned} & \left(\left[i\gamma^0 \frac{\partial}{\partial t} - i\gamma^i \nabla^i - m \right] \psi(t, \mathbf{x}) \right)^* = 0 \Rightarrow \left(-i\gamma^{0*} \frac{\partial}{\partial t} + i\gamma^{i*} \nabla^i - m \right) \psi^*(t, \mathbf{x}) = 0 \\ & \Rightarrow \left(-i\gamma^{0*} \frac{\partial}{\partial t} - i(-\gamma^{i*}) \nabla^i - m \right) \psi^*(t, \mathbf{x}) = 0 \stackrel{\substack{\text{左から} \\ U_T \text{を乗じて}}}{\Rightarrow} U_T \left(-i\gamma^{0*} \frac{\partial}{\partial t} - i(-\gamma^{i*}) \nabla^i - m \right) U_T^{-1} U_T \psi^*(t, \mathbf{x}) = 0 \\ & \stackrel{\text{比較}}{\Rightarrow} \left(-i\gamma^0 \frac{\partial}{\partial t} - i\gamma^i \nabla^i - m \right) \psi(-t, \mathbf{x}) = 0 \Leftrightarrow U_T \left(-i\gamma^{0*} \frac{\partial}{\partial t} - i(-\gamma^{i*}) \nabla^i - m \right) U_T^{-1} U_T \psi^*(t, \mathbf{x}) = 0 \end{aligned}$$

の比較より · · ·

$$\begin{aligned} \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} &\equiv \psi(-t, \mathbf{x}) = U_T \psi^*(x), \quad \gamma^0 = U_T \gamma^{0*} U_T^{-1}, \quad \gamma^i = -U_T \gamma^{i*} U_T^{-1} \\ \mathcal{U}_T \psi^*(x) \mathcal{U}_T^{-1} &\equiv \psi^*(-t, \mathbf{x}) = \eta U_T^{-1} \psi(x) \quad (\eta \psi(x) = U_T \psi^*(-t, \mathbf{x})) \quad (*) \quad \eta = \pm 1 \end{aligned}$$

$\gamma^0 = U_T \gamma^{0*} U_T^{-1}$, $\gamma^i = -U_T \gamma^{i*} U_T^{-1}$ を調べるため

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \boldsymbol{\sigma}^i \\ -\boldsymbol{\sigma}^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \text{ with } \gamma^0 \gamma^0 = -\gamma^i \gamma^i \quad (i: \text{not summed}) = 1$$

を用いると · · ·

$$\gamma^{0*} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \gamma^0, \quad \gamma^{1,3*} = \begin{pmatrix} 0 & \boldsymbol{\sigma}^{1,3*} \\ -\boldsymbol{\sigma}^{1,3*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma}^{1,3} \\ -\boldsymbol{\sigma}^{1,3} & 0 \end{pmatrix} = \gamma^{1,3}, \quad \gamma^{2*} = \begin{pmatrix} 0 & \boldsymbol{\sigma}^{2*} \\ -\boldsymbol{\sigma}^{2*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & 0 \end{pmatrix} = -\gamma^2$$

これより、

$$\begin{aligned} \gamma^{0*} = \gamma^0 &\Leftarrow \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^0 (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^0 = (-)^2 \gamma^0 = \gamma^0 \\ \gamma^{1,3*} = \gamma^{1,3} &\Leftarrow -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^{1,3} (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) (-\gamma^{1,3}) = \gamma^{1,3} \\ \gamma^{2*} = -\gamma^2 &\Leftarrow -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^2 (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^2 = -(-)^2 \gamma^2 = -\gamma^2 \\ \Rightarrow \gamma^{0*} = \gamma^0 &= \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^0 (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = U_T^{-1} \gamma^0 U_T \Rightarrow U_T = \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3, U_T^{-1} = \eta_\gamma^{-1} \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \\ \gamma^{1,3*} = \gamma^{1,3} &= -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^{1,3} (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = -U_T^{-1} \gamma^{1,3} U_T \Rightarrow U_T = \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3, U_T^{-1} = \eta_\gamma^{-1} \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \\ \gamma^{2*} = -\gamma^2 &= -\boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^2 (\boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3) = -U_T^{-1} \gamma^2 U_T \Rightarrow U_T = \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3, U_T^{-1} = \eta_\gamma^{-1} \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \end{aligned}$$

従って、

$$\begin{aligned} U_T &= \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3 = \eta_\gamma \begin{pmatrix} 0 & \boldsymbol{\sigma}^1 \\ -\boldsymbol{\sigma}^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma}^3 \\ -\boldsymbol{\sigma}^3 & 0 \end{pmatrix} = \eta_\gamma \begin{pmatrix} -\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^3 & 0 \\ 0 & -\boldsymbol{\sigma}^1 \boldsymbol{\sigma}^3 \end{pmatrix} = \eta_\gamma \begin{pmatrix} i\boldsymbol{\sigma}^2 & 0 \\ 0 & i\boldsymbol{\sigma}^2 \end{pmatrix} \Leftarrow \text{通常}, \eta_\gamma = 1 \\ U_T^\dagger &= -\eta_\gamma^* \begin{pmatrix} i\boldsymbol{\sigma}^2 & 0 \\ 0 & i\boldsymbol{\sigma}^2 \end{pmatrix} = U_T^{-1}, \quad U_T^T = -U_T, \quad U_T^* = U_T, \quad U_T U_T^* = -1 \\ \gamma^0 &= U_T \gamma^{0*} U_T^{-1}, \quad \gamma^i = -U_T \gamma^{i*} U_T^{-1} \Rightarrow \gamma^{0\dagger} = \gamma^0 = U_T \gamma^{0T} U_T^{-1}, \quad \gamma^{i\dagger} = \gamma^0 \boldsymbol{\gamma}^i \boldsymbol{\gamma}^0 = -\gamma^i = -U_T \gamma^{iT} U_T^{-1} \\ \Rightarrow \boxed{\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}} \end{aligned}$$

以上から、

$$\boxed{\mathcal{U}_T \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} \equiv \psi(-t, \mathbf{x}) = U_T \psi^*(t, \mathbf{x}) \Leftarrow \gamma^\mu = U_T \gamma^{\mu T} U_T^{-1} \text{ with } U_T = \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3, U_T U_T^* = -1}$$

量子力学の場合と同じ関係式なので、 $\psi(t, \mathbf{x}) = \int d\mathbf{p} \varphi(\mathbf{p}) \exp(-iEt + i\mathbf{p}\mathbf{x})$ に対して

$$\boxed{\mathcal{U}_T \varphi(\mathbf{p}) \mathcal{U}_T^{-1} \equiv \varphi(-\mathbf{p}) = U_T \varphi^*(\mathbf{p})} \text{ with } U_T U_T^* = -1, \eta = -1$$

である。そこで、 $\varphi(-\mathbf{p}) = U_T \varphi^*(\mathbf{p})$ に従って、粒子・反粒子の消滅・生成演算子に対して

$$\begin{aligned} \mathcal{U}_T a^{(\lambda)}(\mathbf{p}) \mathcal{U}_T^{-1} &\equiv a^{(\lambda)}(-\mathbf{p}) = \eta_T^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p}), \mathcal{U}_T b^{(\lambda)\dagger}(\mathbf{p}) \mathcal{U}_T^{-1} \equiv b^{(\lambda)\dagger}(-\mathbf{p}) = \eta_T'^{(\lambda)} b^{(\lambda)\dagger}(\mathbf{p}) \quad (\lambda=1,2) \\ &\Leftrightarrow |\eta_T^{(\lambda)}|^2 = |\eta_T'^{(\lambda)}|^2 = 1 \end{aligned}$$

とする。ここで、内部空間 ($\alpha=1,2,3,4$) には依存しないので $U_T = 1$ 。

ディラック方程式の解の振る舞い

$\varphi(-\mathbf{p}) = U \varphi^*(\mathbf{p})$ を求めてある解を用いて調べてみる (x, y は規格化因子で運動量に依存) :

$$\begin{aligned} U_T \varphi^{(1)*}(\mathbf{p}) &= U_T \begin{pmatrix} Z^* \\ \frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} = \eta_\gamma \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \begin{pmatrix} Z^* \\ \frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} \text{ with } Z = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ &= \eta_\gamma \begin{pmatrix} i\sigma^2 Z^* \\ \frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x^* = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^* \\ &= \eta_\gamma \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} \end{pmatrix} x^* \Leftrightarrow \varphi^{(1)}(-\mathbf{p}) = y \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix} \end{pmatrix}_{\mathbf{p} \rightarrow -\mathbf{p}} = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \end{pmatrix} \\ \Rightarrow U_T \varphi^{(1)*}(\mathbf{p}) &= -\eta_\gamma \varphi^{(1)}(-\mathbf{p}) \end{aligned}$$

$$\begin{aligned} U_T \varphi^{(2)*}(\mathbf{p}) &= U_T \begin{pmatrix} Z^* \\ -\frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} = \eta_\gamma \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \begin{pmatrix} Z^* \\ -\frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} \text{ with } Z = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\ &= \eta_\gamma \begin{pmatrix} i\sigma^2 Z^* \\ -\frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} y^* = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* \end{aligned}$$

$$= \eta_\gamma \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \end{pmatrix} y^* \Leftrightarrow \varphi^{(2)}(-\mathbf{p}) = x \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \end{pmatrix}_{p \rightarrow -p} = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \end{pmatrix}$$

$$\Rightarrow U_T \varphi^{(2)*}(\mathbf{p}) = \eta_\gamma \varphi^{(2)}(-\mathbf{p})$$

$$U_T \varphi^{(3)*}(\mathbf{p}) = U_T \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} Z^* \\ Z^* \end{pmatrix} = \eta_\gamma \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} Z^* \\ Z^* \end{pmatrix} \text{ with } Z = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix}$$

$$= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} i\sigma^2 Z^* \\ i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x^* = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^*$$

$$= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^* \Leftrightarrow \varphi^{(3)}(-\mathbf{p}) = y \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix} \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix}_{p \rightarrow -p} = y \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\ -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix}$$

$$\Rightarrow U_T \varphi^{(3)*}(\mathbf{p}) = -\eta_\gamma \varphi^{(3)}(-\mathbf{p})$$

$$U_T \varphi^{(4)*}(\mathbf{p}) = U_T \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} Z^* \\ Z^* \end{pmatrix} = \eta_\gamma \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} Z^* \\ Z^* \end{pmatrix} \text{ with } Z = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix}$$

$$= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} i\sigma^2 Z^* \\ i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^*$$

$$= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* \Leftrightarrow \varphi^{(4)}(-\mathbf{p}) = x \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix}_{p \rightarrow -p} = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix}$$

$$\Rightarrow U_T \varphi^{(4)*}(\mathbf{p}) = \eta_\gamma \varphi^{(4)}(-\mathbf{p})$$

以上から、

$$U_T \varphi^{(1)*}(\mathbf{p}) = -\eta_\gamma \varphi^{(1)}(-\mathbf{p}), U_T \varphi^{(2)*}(\mathbf{p}) = \eta_\gamma \varphi^{(2)}(-\mathbf{p}), U_T \varphi^{(3)*}(\mathbf{p}) = -\eta_\gamma \varphi^{(3)}(-\mathbf{p}), U_T \varphi^{(4)*}(\mathbf{p}) = \eta_\gamma \varphi^{(4)}(-\mathbf{p})$$

$$\Rightarrow \boxed{U_T \eta_\gamma^{(\lambda)*} \varphi^{(\lambda)*}(\mathbf{p}) = \varphi^{(\lambda)}(-\mathbf{p}) \text{ with } \eta_\gamma^{(\lambda)*} = (-\eta_\gamma^*, \eta_\gamma^*, -\eta_\gamma^*, \eta_\gamma^*)} \text{ for } |\eta_\gamma|^2 = 1$$

ディラック場の時間反転

そこで

$$\psi(x) \sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx)] \quad (xp = Et - \mathbf{p}x)$$

これを使って、時間反転をすれば・・・

$$\begin{aligned} \mathcal{U}_T \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} &\equiv \psi(-t, \mathbf{x}) \\ &= \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-iE(-t) + ip\mathbf{x}) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(iE(-t) - ip\mathbf{x})] \\ &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(iEt + ip\mathbf{x}) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(-iEt - ip\mathbf{x})] \\ &= \int_{-\infty}^{-\infty} \frac{d(-\mathbf{p})}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) \exp(iEt + i(-\mathbf{p})\mathbf{x}) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(-\mathbf{p}) \exp(iEt - i(-\mathbf{p})\mathbf{x})] \\ &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)}(-\mathbf{p}) a^{(\lambda)}(-\mathbf{p}) \exp(iEt - ip\mathbf{x}) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(-\mathbf{p}) \exp(-iEt + ip\mathbf{x})] \\ &\Leftarrow a^{(\lambda)}(-\mathbf{p}) = \eta_T^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p}), b^{(\lambda)\dagger}(-\mathbf{p}) = \eta_T'^{(\lambda)} b^{(\lambda)\dagger}(\mathbf{p}) \quad (\lambda = 1, 2) \Leftarrow |\eta_T^{(\lambda)}|^2 = |\eta_T'^{(\lambda)}|^2 = 1 \\ &\Leftarrow \eta_\gamma^{(\lambda)*} U_T \varphi^{(\lambda)*}(\mathbf{p}) = \varphi^{(\lambda)}(-\mathbf{p}) \text{ with } \eta_\gamma^{(\lambda)*} = (-\eta_\gamma^*, \eta_\gamma^*, -\eta_\gamma^*, \eta_\gamma^*) \Leftarrow |\eta_\gamma^{(\lambda)}|^2 = 1 \\ &= \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\eta_\gamma^{(\lambda)*} U_T \varphi^{(\lambda)*}(\mathbf{p}) \eta_T^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p}) \exp(iEt - ip\mathbf{x}) + \eta_\gamma^{(\lambda+2)*} U_T \varphi^{(\lambda+2)*}(-\mathbf{p}) \eta_T'^{(\lambda)} b^{(\lambda)\dagger}(\mathbf{p}) \exp(-iEt + ip\mathbf{x})] \end{aligned}$$

これが、

$$\begin{aligned} \psi^*(x) &\sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)*}(\mathbf{p}) a^{(\lambda)*}(\mathbf{p}) \exp(ipx) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)T}(\mathbf{p}) \exp(-ipx)] \\ &\Leftarrow a^{(\lambda)T} = a^{(\lambda)} \leftrightarrow a^{(\lambda)*} = a^{(\lambda)\dagger} \text{ 等} \\ &= \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 [\varphi^{(\lambda)*}(\mathbf{p}) a^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)}(\mathbf{p}) \exp(-ipx)] \end{aligned}$$

で表されるには、 $\eta_\gamma^{(\lambda)*} \eta_T^{(\lambda)} = \eta_\gamma^{(\lambda+2)*} \eta_T'^{(\lambda)} = \kappa$ ($\lambda = 1, 2$) が必要になる。

$$\begin{aligned} \eta_\gamma^{(1)*} \eta_T^{(1)} &= \eta_\gamma^{(2)*} \eta_T^{(2)} = \eta_\gamma^{(3)*} \eta_T'^{(1)} = \eta_\gamma^{(4)*} \eta_T'^{(2)} = \kappa \Leftarrow \eta_\gamma^{(\lambda)*} = (-\eta_\gamma^*, \eta_\gamma^*, -\eta_\gamma^*, \eta_\gamma^*) \quad (\eta_\gamma = 1: \text{標準}) \\ -\eta_\gamma^* \eta_T^{(1)} &= \eta_\gamma^* \eta_T^{(2)} = -\eta_\gamma^* \eta_T'^{(1)} = \eta_\gamma^* \eta_T'^{(2)} = \kappa \Rightarrow -\eta_T^{(1)} = \eta_T^{(2)} = -\eta_T'^{(1)} = \eta_T'^{(2)} \\ (\eta_T^{(1)}, \eta_T^{(2)}, -\eta_T'^{(1)}, \eta_T'^{(2)}) &= (\eta_T, -\eta_T, \eta_T, -\eta_T) \quad \& \quad \kappa = -\eta_T \end{aligned}$$

結局、

$$\boxed{\begin{aligned} a^{(1)}(-\mathbf{p}) &= \eta_T a^{(1)\dagger}(\mathbf{p}), & a^{(2)}(-\mathbf{p}) &= -\eta_T a^{(2)\dagger}(\mathbf{p}), & b^{(1)\dagger}(-\mathbf{p}) &= \eta_T b^{(1)}(\mathbf{p}), & b^{(2)\dagger}(-\mathbf{p}) &= -\eta_T b^{(2)}(\mathbf{p}) \\ \varphi^{(1)}(-\mathbf{p}) &= -\eta_\gamma^* U_T \varphi^{(1)*}(\mathbf{p}), & \varphi^{(2)}(-\mathbf{p}) &= \eta_\gamma^* U_T \varphi^{(2)*}(\mathbf{p}), & \varphi^{(3)}(-\mathbf{p}) &= -\eta_\gamma^* U_T \varphi^{(3)*}(\mathbf{p}), & \varphi^{(4)}(-\mathbf{p}) &= \eta_\gamma^* U_T \varphi^{(4)*}(\mathbf{p}) \end{aligned}}$$

従って、時間反転は、

$$\begin{aligned} \psi(-t, \mathbf{x}) &\sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\eta_\gamma^{(\lambda)*} U_T \varphi^{(\lambda)*}(\mathbf{p}) \eta_T^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) + \eta_\gamma^{(\lambda+2)*} U_T \varphi^{(\lambda+2)*}(-\mathbf{p}) \eta_T^{(\lambda)\dagger}(\mathbf{p}) \exp(-ipx) \right] \\ &= \kappa U_T \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi^{(\lambda)*}(\mathbf{p}) a^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) + \varphi^{(\lambda+2)*}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(-ipx) \right] \\ &= \kappa U_T \psi^*(t, \mathbf{x}) = -\eta_T U_T \psi^*(t, \mathbf{x}) = -\eta_T U_T \gamma^{0T} [\psi^\dagger(t, \mathbf{x}) \gamma^0]^T \stackrel{\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}}{=} -\eta_T \gamma^0 U_T \overline{\psi(t, \mathbf{x})}^T \\ \overline{\psi(-t, \mathbf{x})} &= \psi^\dagger(-t, \mathbf{x}) \gamma^0 = -\eta_T^* \psi^T(t, \mathbf{x}) U_T^\dagger \gamma^0 \end{aligned}$$

になる。

$$\boxed{\mathcal{U}_T \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} \equiv \psi(-t, \mathbf{x}) = -\eta_T \gamma^0 U_T \overline{\psi(t, \mathbf{x})}^T, \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \mathcal{U}_T^{-1} \equiv \overline{\psi(-t, \mathbf{x})} = -\eta_T^* \psi^T(t, \mathbf{x}) U_T^\dagger \gamma^0}$$

場の順序

演算子の順序への影響として、具体的に粒子の散乱を考える：

$$\text{Amp}(t = -\infty \rightarrow t = \infty) \sim b^\dagger(\mathbf{q}') a^\dagger(\mathbf{p}') b(\mathbf{q}) a(\mathbf{p})$$

この振幅 (Amplitude) は、

$$\text{粒子}(\mathbf{p}) + \text{反粒子}(\mathbf{q}) \rightarrow \text{粒子}(\mathbf{p}') + \text{反粒子}(\mathbf{q}')$$

の散乱を表している。この反応を時間を逆にして眺めると、進行方向を逆にして

$$\text{粒子}(-\mathbf{p}') + \text{反粒子}(-\mathbf{q}') \rightarrow \text{粒子}(-\mathbf{p}) + \text{反粒子}(-\mathbf{q})$$

のように散乱が起こるはずである。この散乱に対応する振幅は、

$$\text{Amp}(t = \infty \rightarrow t = -\infty) \sim a^\dagger(-\mathbf{p}) b^\dagger(-\mathbf{q}) a(-\mathbf{p}') b(-\mathbf{q}')$$

である。ところで、時間反転に対して

$$\text{粒子} : \mathcal{U}_T a(\mathbf{p}) \mathcal{U}_T^{-1} = \eta_T a^\dagger(-\mathbf{p}), \text{ 反粒子} : \mathcal{U}_T b(\mathbf{p}) \mathcal{U}_T^{-1} = \eta_T^* b^\dagger(-\mathbf{p})$$

の変換をするので

$$\begin{aligned} \mathcal{U}_T \text{Amp}(t = \infty \rightarrow t = -\infty) \mathcal{U}_T^{-1} &\sim \mathcal{U}_T b^\dagger(\mathbf{q}') a^\dagger(\mathbf{p}') b(\mathbf{q}) a(\mathbf{p}) \mathcal{U}_T^{-1} \\ &= \eta_T^* b(-\mathbf{q}') \eta_T^* a(-\mathbf{p}') \eta_T b^\dagger(-\mathbf{q}) \eta_T a^\dagger(-\mathbf{p}) \\ &= b(-\mathbf{q}') a(-\mathbf{p}') b^\dagger(-\mathbf{q}) a^\dagger(-\mathbf{p}) \end{aligned}$$

になる。実際の過程とは、順序が異なる。従って、時間反転では、

● 演算子の順序を逆にする

操作を同時に行う必要がある。つまり、

$$\begin{aligned} \mathcal{U}_T \text{Amp}(t = \infty \rightarrow t = -\infty) \mathcal{U}_T^{-1} &\sim \mathcal{U}_T b^\dagger(\mathbf{q}') a^\dagger(\mathbf{p}') b(\mathbf{q}) a(\mathbf{p}) \mathcal{U}_T^{-1} \\ &= \underset{\text{順序を逆に}}{\left(\mathcal{U}_T a(\mathbf{p}) \mathcal{U}_T^{-1} \right)} \left(\mathcal{U}_T b(\mathbf{q}) \mathcal{U}_T^{-1} \right) \left(\mathcal{U}_T a^\dagger(\mathbf{p}') \mathcal{U}_T^{-1} \right) \left(\mathcal{U}_T b^\dagger(\mathbf{q}') \mathcal{U}_T^{-1} \right) \end{aligned}$$

である。

従って、 x -表示では、スピン $1/2$ の場合に

$$\mathcal{U}_T \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} = -\eta_T \gamma^0 U_T \overline{\psi(t, \mathbf{x})}^T, \quad \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \mathcal{U}_T^{-1} = -\eta_T^* \psi^T(t, \mathbf{x}) U_T^\dagger \gamma^0$$

なので、例えば、スカラー量では・・・

$$\begin{aligned} \mathcal{U}_T \overline{\psi_1(x)} \psi_2(x) \cdots \overline{\psi_{2N-1}(x)} \psi_{2N}(x) \mathcal{U}_T^{-1} &\stackrel{\text{順序逆}}{=} \mathcal{U}_T \psi_{2N}^T(x) \mathcal{U}_T^{-1} \mathcal{U}_T \overline{\psi_{2N-1}(x)}^T \mathcal{U}_T^{-1} \cdots \mathcal{U}_T \psi_2^T(x) \mathcal{U}_T^{-1} \mathcal{U}_T \overline{\psi_1(x)}^T \mathcal{U}_T^{-1} \\ &= \left[-\eta_T \gamma^0 U_T \overline{\psi_{2N}(t, \mathbf{x})}^T \right]^T \left[-\eta_T^* \psi_{2N-1}^T(t, \mathbf{x}) U_T^\dagger \gamma^0 \right]^T \cdots \left[-\eta_T \gamma^0 U_T \overline{\psi_2(t, \mathbf{x})}^T \right]^T \left[-\eta_T^* \psi_1^T(t, \mathbf{x}) U_T^\dagger \gamma^0 \right]^T \\ &= \overline{\psi_{2N}(t, \mathbf{x})} (-U_T) \gamma^{0T} \gamma^{0T} (-U_T^\dagger) \psi_{2N-1}(t, \mathbf{x}) \cdots \overline{\psi_2(t, \mathbf{x})} (-U_T) \gamma^{0T} \gamma^{0T} (-U_T^\dagger) \psi_1(t, \mathbf{x}) \\ &= \overline{\psi_{2N}(t, \mathbf{x})} \psi_{2N-1}(t, \mathbf{x}) \cdots \overline{\psi_2(t, \mathbf{x})} \psi_1(t, \mathbf{x}) = \psi_{2N}^\dagger(t, \mathbf{x}) \gamma^0 \psi_{2N-1}(t, \mathbf{x}) \cdots \psi_2^\dagger(t, \mathbf{x}) \gamma^0 \psi_1(t, \mathbf{x}) \\ &= [\psi_1^\dagger(t, \mathbf{x}) \gamma^{0\dagger} \psi_2(t, \mathbf{x}) \cdots \psi_{2N-1}^\dagger(t, \mathbf{x}) \gamma^{0\dagger} \psi_{2N}(t, \mathbf{x})]^\dagger = [\overline{\psi_1(x)} \psi_2(x) \cdots \overline{\psi_{2N-1}(x)} \psi_{2N}(x)]^\dagger \end{aligned}$$

つまり、大雑把にいって「時間反転はエルミート共役操作」である（スカラー量以外は自明でない）。

第三節：粒子・反粒子反転（C変換）

スピン 0

粒子・反粒子の消滅演算子

$$\text{粒子} : a(\mathbf{p}), a^\dagger(\mathbf{p}), \text{ 反粒子} : b(\mathbf{p}), b^\dagger(\mathbf{p})$$

の入れ替えによる：

$$\begin{aligned} a^C(\mathbf{p}) &= \mathcal{U}_C a(\mathbf{p}) \mathcal{U}_C^\dagger = \eta_C b(\mathbf{p}), \quad b^{C\dagger}(\mathbf{p}) = \mathcal{U}_C b^\dagger(\mathbf{p}) \mathcal{U}_C^\dagger = \eta_C a^\dagger(\mathbf{p}) \\ a^{C\dagger}(\mathbf{p}) &= \mathcal{U}_C a^\dagger(\mathbf{p}) \mathcal{U}_C^\dagger = \eta_C^* b^\dagger(\mathbf{p}), \quad b^C(\mathbf{p}) = \mathcal{U}_C b(\mathbf{p}) \mathcal{U}_C^\dagger = \eta_C^* a(\mathbf{p}) \end{aligned}$$

この場合は、

$$\begin{aligned} \phi^C(x) &= \mathcal{U}_C \phi(x) \mathcal{U}_C^\dagger \sim \mathcal{U}_C \int \frac{d\mathbf{p}}{2E} [a(\mathbf{p}) \exp(-ipx) + b^\dagger(\mathbf{p}) \exp(ipx)] \mathcal{U}_C^\dagger \\ &= \int \frac{d\mathbf{p}}{2E} [\eta_C b(\mathbf{p}) \exp(-ipx) + \eta_C a^\dagger(\mathbf{p}) \exp(ipx)] = \eta_C \phi^\dagger(x) \end{aligned}$$

より、

$$\boxed{\phi^C(x) = \mathcal{U}_C \phi(x) \mathcal{U}_C^\dagger = \eta_C \phi^\dagger(x), \quad \phi^C(x) = \mathcal{U}_C \phi^\dagger(x) \mathcal{U}_C^\dagger = \eta_C^* \phi(x)}$$

である。

荷電反転パリティ

粒子反粒子の 2 体系は・・・

$$|\text{系}\rangle = \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle$$

これに粒子反粒子反転を行うと、

$$\begin{aligned} |\text{系}^c\rangle &= U_c |\text{系}\rangle = \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) U_c a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle \\ &= \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) U_c a^\dagger(\mathbf{p}) U_c^{-1} U_c b^\dagger(-\mathbf{p}) U_c^{-1} U_c |0\rangle \stackrel{U_c|0\rangle=|0\rangle}{=} \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) U_c a^\dagger(\mathbf{p}) U_c^{-1} U_c b^\dagger(-\mathbf{p}) U_c^{-1} |0\rangle \\ &= \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) (\eta_c^* b^\dagger(\mathbf{p})) (\eta_c a^\dagger(-\mathbf{p})) |0\rangle \stackrel{|\eta_c|^2=1}{=} \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} f(\mathbf{p}) b^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p}) |0\rangle \\ &= \int_{-\infty}^{\infty} \frac{d(-\mathbf{p})}{2E} f(-\mathbf{p}) b^\dagger(-\mathbf{p}) a^\dagger(\mathbf{p}) |0\rangle \stackrel{[a^\dagger(\mathbf{p}), b^\dagger(-\mathbf{p})]=0}{=} \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} f(-\mathbf{p}) a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle \\ &= \int \frac{d\mathbf{p}}{2E} f(-\mathbf{p}) a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle \end{aligned}$$

なので、

$$|\text{系}\rangle = \int \frac{d\mathbf{p}}{2E} f(\mathbf{p}) a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle \quad \& \quad |\text{系}^c\rangle = \int \frac{d\mathbf{p}}{2E} f(-\mathbf{p}) a^\dagger(\mathbf{p}) b^\dagger(-\mathbf{p}) |0\rangle$$

になる。そこで、 U_c の固有値を C とすると、 $U_c |\text{系}\rangle = C |\text{系}\rangle$ 。今の場合、

$$\begin{aligned} |\text{系}^c\rangle &= U_c |\text{系}\rangle = +|\text{系}\rangle \text{ if } f(\mathbf{p}) = f(-\mathbf{p}) \Rightarrow C = +1 \\ |\text{系}^c\rangle &= U_c |\text{系}\rangle = -|\text{系}\rangle \text{ if } f(\mathbf{p}) = -f(-\mathbf{p}) \Rightarrow C = -1 \end{aligned}$$

になる。

スピン 1 (光子)

粒子の生成・消滅演算子は粒子・反粒子変換により粒子に戻るので

$$a^{(\lambda)C}(\mathbf{p}) = \mathcal{U}_c a^{(\lambda)}(\mathbf{p}) \mathcal{U}_c^{-1} = \eta_c a^{(\lambda)}(\mathbf{p}), a_\mu^C(\mathbf{p}) = \mathcal{U}_c a_\mu(\mathbf{p}) \mathcal{U}_c^{-1} = \eta_c a_\mu(\mathbf{p}) \Leftarrow |\eta_c|^2 = 1$$

$$A_\mu^C(x) = \mathcal{U}_c A_\mu(x) \mathcal{U}_c^{-1} \sim \mathcal{U}_c \int_{-\infty}^{\infty} \frac{d\mathbf{p}}{2E} [a_\mu(\mathbf{p}) \exp(-ipx) + a_\mu^\dagger(\mathbf{p}) \exp(ipx)] \mathcal{U}_c^{-1} = \eta_c A_\mu(x)$$

$$A_\mu^C(x) = \mathcal{U}_c A_\mu(x) \mathcal{U}_c^{-1} = \eta_c A_\mu(x)$$

電磁相互作用

電磁相互作用は、ゲージ相互作用で記述され、スピン 0 の $\phi(x)$ に対して

$$\left| (i\partial_\mu + eA_\mu) \phi(x) \right|^2 \Rightarrow i \left(\phi^\dagger(x) \partial^\mu \phi(x) - (\partial^\mu \phi^\dagger(x)) \phi(x) \right) A_\mu(x) \equiv I_{\text{int}}(x)$$

で特徴付けられている。これに、粒子・反粒子変換をすると

$$\begin{aligned} \mathcal{U}_c I_{\text{int}}(x) \mathcal{U}_c^\dagger &= \mathcal{U}_c i \left(\phi^\dagger(x) \partial^\mu \phi(x) - (\partial^\mu \phi^\dagger(x)) \phi(x) \right) A_\mu(x) \mathcal{U}_c^\dagger \\ &= i \left((\eta_c^* \phi(x)) \partial^\mu (\eta_c \phi^\dagger(x)) - (\partial^\mu \eta_c^* \phi(x)) (\eta_c \phi^\dagger(x)) \right) \eta_c^A A_\mu(x) \Leftarrow |\eta_c|^2 = 1 \\ &= i \eta_c^A \left(\phi(x) \partial^\mu \phi^\dagger(x) - (\partial^\mu \phi(x)) \phi^\dagger(x) \right) A_\mu(x) = -i \eta_c^A \left(\phi^\dagger(x) \partial^\mu \phi(x) - (\partial^\mu \phi^\dagger(x)) \phi(x) \right) A_\mu(x) \end{aligned}$$

$$= -\eta_C^A I_{\text{int}}(x) \quad (\text{ここで、} A_\mu(x) \text{の} \eta_C \text{を} \eta_C^A \text{とした})$$

従って、もし、この電磁相互作用が粒子・反粒子変換で不变であるとすると、

$$\mathcal{U}_C I_{\text{int}}(x) \mathcal{U}_C^\dagger = -\eta_C^A I_{\text{int}}(x) = I_{\text{int}}(x) \Rightarrow [\eta_C^A = -1]$$

になる。つまり、光子の粒子・反粒子変換パリティは-1になる。

スピン 1/2 ディラック方程式での粒子・反粒子反転

ディラック方程式を調べると・・・

$$\begin{aligned} & (i\partial_\mu \gamma^\mu - m)_{\alpha\beta} \psi_\beta(x) = (i\partial_\mu \gamma^\mu - m) \psi(x) = 0 \\ & \stackrel{\dagger\text{-操作}}{\Rightarrow} [(i\partial_\mu \gamma^\mu - m) \psi(x)]^\dagger = \psi^\dagger(x) (-i\overline{\partial}_\mu \gamma^{\mu\dagger} - m) \stackrel{\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0}{=} \psi^\dagger(x) (-i\overline{\partial}_\mu \gamma^0 \gamma^\mu \gamma^0 - m) \\ & \stackrel{\overline{\psi(x)}=\psi^\dagger(t,x)\gamma^0}{=} \overline{\psi(x)} (-i\overline{\partial}_\mu \gamma^\mu \gamma^0 - \gamma^0 m) = \overline{\psi(x)} (-i\overline{\partial}_\mu \gamma^\mu - m) \gamma^0 = 0 \\ & \Rightarrow \overline{\psi(x)} (-i\overline{\partial}_\mu \gamma^\mu - m) = 0 \stackrel{T\text{-操作}}{\Rightarrow} [\overline{\psi(x)} (-i\overline{\partial}_\mu \gamma^\mu - m)]^T = (-i\partial_\mu \gamma^{\mu T} - m) [\overline{\psi(x)}]^T = 0 \\ & \stackrel{\text{Unitary演算子}}{\Rightarrow} U_C (-i\partial_\mu \gamma^{\mu T} - m) U_C^{-1} U_C [\overline{\psi(x)}]^T = 0 \Leftarrow \text{任意性の導入} \\ & \qquad \qquad \qquad = (i\partial_\mu (-U_C \gamma^{\mu T} U_C^{-1}) - m) U_C [\overline{\psi(x)}]^T = 0 \\ & \stackrel{\text{比較}}{\Rightarrow} (i\partial_\mu (-U_C \gamma^{\mu T} U_C^{-1}) - m) U_C [\overline{\psi(x)}]^T = 0 \Leftrightarrow (i\partial_\mu \gamma^\mu - m) \psi(x) = 0 \end{aligned}$$

と比較して

$$\boxed{\gamma^\mu = -U_C \gamma^{\mu T} U_C^{-1} \& \psi^C(x) \equiv U_C [\overline{\psi(x)}]^T} \Rightarrow \boxed{(i\partial_\mu \gamma^\mu - m) \psi^C(x) = 0}$$

になる。これを、反粒子のディラック方程式という。時間反転で求めた：

$$\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1} \quad \text{with } U_T = \eta_\gamma \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3, U_T^{-1} = \eta_\gamma^{-1} \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1$$

を用いて、

$$\gamma_5 \gamma^\mu \gamma_5 = \gamma_5 U_T \gamma^{\mu T} U_T^{-1} \gamma_5 = \gamma_5 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3 \gamma^{\mu T} \gamma_5 \boldsymbol{\gamma}^3 \boldsymbol{\gamma}^1 \Rightarrow \gamma^\mu = -U_C \gamma^{\mu T} U_C^{-1} \text{ and } U_C = \eta'_\gamma \gamma_5 U_T$$

がわかる。従って、

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \boldsymbol{\gamma}^i = \begin{pmatrix} 0 & \boldsymbol{\sigma}^i \\ -\boldsymbol{\sigma}^i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \text{ with } \gamma^0 \gamma^0 = -\boldsymbol{\gamma}^i \boldsymbol{\gamma}^i \quad (i: \text{not summed}) = 1 \\ U_C &= \eta'_\gamma \gamma_5 U_T = \eta'_\gamma (\eta_\gamma \gamma_5 \boldsymbol{\gamma}^1 \boldsymbol{\gamma}^3)^{\eta'_\gamma \eta_\gamma |\eta_\gamma|^2 = 1} = \eta_\gamma \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \begin{pmatrix} i\boldsymbol{\sigma}^2 & 0 \\ 0 & i\boldsymbol{\sigma}^2 \end{pmatrix} = \eta_\gamma \begin{pmatrix} 0 & i\boldsymbol{\sigma}^2 \\ i\boldsymbol{\sigma}^2 & 0 \end{pmatrix} \\ U_C^{-1} &\stackrel{\eta_\gamma^{-1} = \eta_\gamma^*}{=} -\eta_\gamma^* \begin{pmatrix} 0 & i\boldsymbol{\sigma}^2 \\ i\boldsymbol{\sigma}^2 & 0 \end{pmatrix} = U_C^\dagger, \quad U_C^T = -U_C \quad (\text{通常, } \eta_\gamma = -1) \end{aligned}$$

運動量空間では・・・

$$\psi(x) = \int d\mathbf{p} \varphi(\mathbf{p}) \exp(-ipx) \Rightarrow \psi^c(x) = U_C \left[\overline{\psi(x)} \right]^T = \int d\mathbf{p} U_C \overline{\varphi(\mathbf{p})}^T \exp(-ipx)$$

より $\varphi^c(\mathbf{p}) = U_C \overline{\varphi(\mathbf{p})}^T$ である。

ディラック方程式の解の振る舞い

$\varphi^c(\mathbf{p}) = U_C \overline{\varphi(\mathbf{p})}^T$ を求めた解を用いて調べてみる (x, y は規格化因子で運動量に依存) :

$$\begin{aligned} U_C \overline{\varphi^{(1)}(\mathbf{p})}^T &= U_C \left[\left(Z^\dagger, \frac{|\mathbf{p}|}{|E|+m} Z^\dagger \right) \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \right]^T = \eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} Z^* \\ -\frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} \text{ with } Z = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ &= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \\ i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x^* = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^* \\ &= \eta_\gamma \begin{pmatrix} -\frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^* \Leftrightarrow \varphi^{(3)}(-\mathbf{p}) = y \begin{pmatrix} -\frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix} \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix}_{p \rightarrow -p} = \begin{pmatrix} -\frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\ -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} y \end{aligned}$$

$$\Rightarrow U_C \overline{\varphi^{(1)}(\mathbf{p})}^T = -\eta_\gamma \varphi^{(3)}(-\mathbf{p})$$

$$\begin{aligned} U_C \overline{\varphi^{(2)}(\mathbf{p})}^T &= U_C \left[\left(Z^\dagger, -\frac{|\mathbf{p}|}{|E|+m} Z^\dagger \right) \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \right]^T = \eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} Z^* \\ \frac{|\mathbf{p}|}{|E|+m} Z^* \end{pmatrix} \text{ with } Z = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\ &= \eta_\gamma \begin{pmatrix} \frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \\ i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} y^* = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* \\ &= \eta_\gamma \begin{pmatrix} \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* \Leftrightarrow \varphi^{(4)}(-\mathbf{p}) = x \begin{pmatrix} \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \\ 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix}_{p \rightarrow -p} = \begin{pmatrix} \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x \end{aligned}$$

$$\Rightarrow U_C \overline{\varphi^{(2)}(\mathbf{p})}^T = \eta_\gamma \varphi^{(4)}(-\mathbf{p})$$

$$\begin{aligned}
U_C \overline{\varphi^{(3)}(\mathbf{p})}^T &= U_C \left[\left(-\frac{|\mathbf{p}|}{|E|+m} Z^\dagger, Z^\dagger \right) \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \right]^T = \eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} |\mathbf{p}| \\ -Z^* \end{pmatrix} \text{ with } Z = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\
&= \eta_\gamma \begin{pmatrix} -i\sigma^2 Z^* \\ -\frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x^* = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -1 \end{pmatrix} x^* \\
&= \eta_\gamma \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \mathbf{p}^1 - i\mathbf{p}^2 \\ |\mathbf{p}| + \mathbf{p}^3 \end{pmatrix} \\ 1 \end{pmatrix} x^* \Leftrightarrow \varphi^{(1)}(-\mathbf{p}) = y \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \mathbf{p}^1 - i\mathbf{p}^2 \\ |\mathbf{p}| - \mathbf{p}^3 \end{pmatrix} \\ 1 \end{pmatrix}_{p \rightarrow -p} = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \mathbf{p}^1 - i\mathbf{p}^2 \\ |\mathbf{p}| + \mathbf{p}^3 \end{pmatrix} \\ 1 \end{pmatrix} \\
\Rightarrow U_C \overline{\varphi^{(3)}(\mathbf{p})}^T &= \eta_\gamma \varphi^{(1)}(-\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
U_C \overline{\varphi^{(4)}(\mathbf{p})}^T &= U_C \left[\left(\frac{|\mathbf{p}|}{|E|+m} Z^\dagger, Z^\dagger \right) \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \right]^T = -\eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} |\mathbf{p}| \\ -Z^* \end{pmatrix} \text{ with } Z = y \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\
&= \eta_\gamma \begin{pmatrix} -i\sigma^2 Z^* \\ \frac{|\mathbf{p}|}{|E|+m} i\sigma^2 Z^* \end{pmatrix} \text{ with } i\sigma^2 Z^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} y^* \\
&= \eta_\gamma \begin{pmatrix} -1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ \frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} \mathbf{p}^1 + i\mathbf{p}^2 \\ |\mathbf{p}| + \mathbf{p}^3 \end{pmatrix} \\ 1 \end{pmatrix} y^* \Leftrightarrow \varphi^{(2)}(-\mathbf{p}) = x \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \\ 1 \end{pmatrix}_{p \rightarrow -p} = x \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E|+m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \end{pmatrix} \\
\Rightarrow U_C \overline{\varphi^{(4)}(\mathbf{p})}^T &= -\eta_\gamma \varphi^{(2)}(-\mathbf{p})
\end{aligned}$$

$$\boxed{
\begin{aligned}
U_C \overline{\varphi^{(1)}(\mathbf{p})}^T &= -\eta_\gamma \varphi^{(3)}(-\mathbf{p}), \quad U_C \overline{\varphi^{(2)}(\mathbf{p})}^T = \eta_\gamma \varphi^{(4)}(-\mathbf{p}) \\
U_C \overline{\varphi^{(3)}(\mathbf{p})}^T &= \eta_\gamma \varphi^{(1)}(-\mathbf{p}), \quad U_C \overline{\varphi^{(4)}(\mathbf{p})}^T = -\eta_\gamma \varphi^{(2)}(-\mathbf{p})
\end{aligned}
}$$

(この表式で、 $\mathbf{p} \rightarrow -\mathbf{p}$ としないように！！！すべてユニタリー変換で与えられる)
これらの表式が無矛盾であることをチェックすると・・・

$$\begin{aligned}
U_C \overline{\varphi^{(1)}(\mathbf{p})}^T &= -\eta_\gamma \varphi^{(3)}(-\mathbf{p}) \Leftarrow \varphi^{(2)}(-\mathbf{p}) = \gamma^0 \varphi^{(1)}(\mathbf{p}), \quad \varphi^{(3)}(-\mathbf{p}) = -\gamma^0 \varphi^{(4)}(\mathbf{p}) \\
U_C \overline{\gamma^0 \varphi^{(2)}(-\mathbf{p})}^T &= -\eta_\gamma (-\gamma^0 \varphi^{(4)}(\mathbf{p})) \Rightarrow U_C \left[(\gamma^0 \varphi^{(2)}(-\mathbf{p}))^\dagger \gamma^0 \right]^T = \eta_\gamma \gamma^0 \varphi^{(4)}(\mathbf{p})
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow U_C \gamma^{0T} \left[\varphi^{(2)\dagger}(-\mathbf{p}) \gamma^{0\dagger} \right]^T = \eta_\gamma \gamma^0 \varphi^{(4)}(\mathbf{p}) \Rightarrow U_C \gamma^{0T} \left[\varphi^{(2)\dagger}(-\mathbf{p}) \gamma^0 \right]^T = \eta_\gamma \gamma^0 \varphi^{(4)}(\mathbf{p}) \\
& \Rightarrow U_C \gamma^{0T} \gamma^{0T} \varphi^{(2)\dagger T}(-\mathbf{p}) = \eta_\gamma \gamma^0 \varphi^{(4)}(\mathbf{p}) \Rightarrow U_C \varphi^{(2)\dagger T}(-\mathbf{p}) = \eta_\gamma \gamma^0 \varphi^{(4)}(\mathbf{p}) \\
& \Rightarrow \varphi^{(2)T}(-\mathbf{p}) U_C^\dagger = \eta_\gamma^* \varphi^{(4)\dagger}(\mathbf{p}) \gamma^{0\dagger} \stackrel{\gamma^{0\dagger} = \gamma^0}{\Rightarrow} \varphi^{(2)T}(-\mathbf{p}) U_C^\dagger = \eta_\gamma^* \varphi^{(4)\dagger}(\mathbf{p}) \gamma^0 \\
& \Rightarrow \varphi^{(2)T}(-\mathbf{p}) U_C^\dagger = \eta_\gamma^* \overline{\varphi^{(4)}(\mathbf{p})} \Rightarrow U_C^{\dagger T} \varphi^{(2)}(-\mathbf{p}) = \eta_\gamma^* \overline{\varphi^{(4)}(\mathbf{p})}^T \\
& \Rightarrow \overline{\varphi^{(4)}(\mathbf{p})}^T \stackrel{|\eta_C|^2=1}{=} \eta_\gamma U_C^{\dagger T} \varphi^{(2)}(-\mathbf{p}) \Rightarrow \overline{\varphi^{(4)}(\mathbf{p})}^T \stackrel{|\eta_C|^2=1}{=} -\eta_\gamma U_C^\dagger \varphi^{(2)}(-\mathbf{p}) \\
& \Rightarrow U_C \overline{\varphi^{(4)}(\mathbf{p})}^T = -\eta_\gamma \varphi^{(2)}(-\mathbf{p}) \leftarrow \text{解から得られた表式と一致} \\
& U_C \overline{\varphi^{(3)}(\mathbf{p})}^T = \eta_\gamma \varphi^{(1)}(-\mathbf{p}) \Leftarrow \varphi^{(4)}(-\mathbf{p}) = -\gamma^0 \varphi^{(3)}(\mathbf{p}), \varphi^{(1)}(-\mathbf{p}) = \gamma^0 \varphi^{(2)}(\mathbf{p}) \\
& -U_C \overline{\gamma^0 \varphi^{(4)}(-\mathbf{p})}^T = \eta_\gamma (\gamma^0 \varphi^{(2)}(\mathbf{p})) \Rightarrow (\varphi^{(2)} \rightarrow \varphi^{(4)}, \eta_\gamma \rightarrow -\eta_\gamma \text{ at the previous case}) \\
& \Rightarrow U_C \overline{\varphi^{(2)}(\mathbf{p})}^T = \eta_\gamma \varphi^{(4)}(-\mathbf{p}) \leftarrow \text{解から得られた表式と一致}
\end{aligned}$$

これ以外も、それぞれ無矛盾であることがわかる。また、一般に

$$\boxed{U_C \overline{A(\mathbf{p})}^T = B(-\mathbf{p})} \Rightarrow \overline{A(\mathbf{p})}^T = U_C^{-1} B(-\mathbf{p}) \Rightarrow \overline{A(\mathbf{p})} = B^T(-\mathbf{p}) U_C^{T-1}$$

$$\Rightarrow A^\dagger(\mathbf{p}) \gamma^0 = B^T(-\mathbf{p})(-U_C^{-1}) \Rightarrow A^\dagger(\mathbf{p}) = -B^T(-\mathbf{p}) U_C^{-1} \gamma^0$$

$$\Rightarrow A(\mathbf{p}) = -\gamma^{0\dagger} U_C B^{T\dagger}(-\mathbf{p}) \Rightarrow A(\mathbf{p}) = -\gamma^0 U_C B^{T\dagger}(-\mathbf{p})$$

$$\Rightarrow A(\mathbf{p}) = -(-U_C \gamma^{0T}) B^{T\dagger}(-\mathbf{p}) = U_C [B^\dagger(-\mathbf{p}) \gamma^0]^T = U_C \overline{B(-\mathbf{p})}^T \Rightarrow \boxed{A(\mathbf{p}) = U_C \overline{B(-\mathbf{p})}^T}$$

である。

ディラック場の粒子・反粒子反転

粒子・反粒子の消滅演算子

$$\text{粒子} : a^{(\lambda)}(\mathbf{p}), a^{(\lambda)\dagger}(\mathbf{p}), \text{反粒子} : b^{(\lambda)}(\mathbf{p}), b^{(\lambda)\dagger}(\mathbf{p}) \quad (\lambda=1,2)$$

の入れ替えを

$$\mathcal{U}_C a^{(\lambda)}(\mathbf{p}) \mathcal{U}_C^{-1} = \eta_C^{(\lambda)} b^{(\lambda)}(\mathbf{p}), \mathcal{U}_C b^{(\lambda)\dagger}(\mathbf{p}) \mathcal{U}_C^{-1} = \eta_C'{}^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p})$$

そこで

$$\begin{aligned}
\psi(x) & \sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-ipx) + \varphi^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(ipx) \right] \\
& = \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \exp(-iEt + ipx) + \varphi^{(\lambda+2)}(-\mathbf{p}) b^{(\lambda)\dagger}(\mathbf{p}) \exp(iEt - ipx) \right]
\end{aligned}$$

これを使って、粒子・反粒子反転をすれば・・・

$$\begin{aligned}
\mathcal{U}_C \psi(x) \mathcal{U}_C^{-1} & \sim \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi^{(\lambda)}(\mathbf{p}) \mathcal{U}_C a^{(\lambda)}(\mathbf{p}) \mathcal{U}_C^{-1} \exp(ipx) + \varphi^{(\lambda+2)}(-\mathbf{p}) \mathcal{U}_C b^{(\lambda)\dagger}(\mathbf{p}) \mathcal{U}_C^{-1} \exp(-ipx) \right] \\
& = \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\varphi^{(\lambda)}(\mathbf{p}) \eta_C^{(\lambda)} b^{(\lambda)}(\mathbf{p}) \exp(ipx) + \varphi^{(\lambda+2)}(-\mathbf{p}) \eta_C'{}^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p}) \exp(-ipx) \right]
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{d\mathbf{p}}{2E} \left[\left(\varphi^{(1)}(\mathbf{p}) \eta_C^{(1)} b^{(1)}(\mathbf{p}) + \varphi^{(2)}(\mathbf{p}) \eta_C^{(2)} b^{(2)}(\mathbf{p}) \right) \exp(ipx) \right. \\
&\quad \left. + \left(\varphi^{(3)}(-\mathbf{p}) \eta_C'^{(1)} a^{(1)\dagger}(\mathbf{p}) + \varphi^{(4)}(-\mathbf{p}) \eta_C'^{(2)} a^{(2)\dagger}(\mathbf{p}) \right) \exp(-ipx) \right] \\
&\Leftarrow U_C \overline{A(\mathbf{p})}^T = B(-\mathbf{p}) \rightarrow A(\mathbf{p}) = U_C \overline{B(-\mathbf{p})}^T \\
&\Leftarrow U_C \overline{\varphi^{(1)}(\mathbf{p})}^T = -\eta_\lambda \varphi^{(3)}(-\mathbf{p}) \rightarrow \varphi^{(1)}(\mathbf{p}) = -\eta_\gamma^* U_C \overline{\varphi^{(3)}(-\mathbf{p})}^T \\
&\Leftarrow U_C \overline{\varphi^{(2)}(\mathbf{p})}^T = \eta_\lambda \varphi^{(4)}(-\mathbf{p}) \rightarrow \varphi^{(2)}(\mathbf{p}) = \eta_\gamma^* U_C \overline{\varphi^{(4)}(-\mathbf{p})}^T \\
&\Leftarrow U_C \overline{\varphi^{(3)}(\mathbf{p})}^T = \eta_\lambda \varphi^{(1)}(-\mathbf{p}) \rightarrow \varphi^{(3)}(\mathbf{p}) = \eta_\gamma^* U_C \overline{\varphi^{(1)}(-\mathbf{p})}^T \\
&\Leftarrow U_C \overline{\varphi^{(4)}(\mathbf{p})}^T = -\eta_\lambda \varphi^{(2)}(-\mathbf{p}) \rightarrow \varphi^{(4)}(\mathbf{p}) = -\eta_\gamma^* U_C \overline{\varphi^{(2)}(-\mathbf{p})}^T \\
&= \int \frac{d\mathbf{p}}{2E} \left[\left(\overline{-\eta_\gamma^* U_C \varphi^{(3)}(-\mathbf{p})}^T \right) \eta_C^{(1)} b^{(1)}(\mathbf{p}) + \left(\overline{\eta_\gamma^* U_C \varphi^{(4)}(-\mathbf{p})}^T \right) \eta_C^{(2)} b^{(2)}(\mathbf{p}) \right] \exp(ipx) \\
&\quad + \left[\left(\overline{-\eta_\gamma^* U_C \varphi^{(1)}(-\mathbf{p})}^T \right) \eta_C'^{(1)} a^{(1)\dagger}(\mathbf{p}) + \left(\overline{\eta_\gamma^* U_C \varphi^{(2)}(-\mathbf{p})}^T \right) \eta_C'^{(2)} a^{(2)\dagger}(\mathbf{p}) \right] \exp(-ipx) \\
&= \eta_\gamma^* U_C \int \frac{d\mathbf{p}}{2E} \left[\left(\overline{-\eta_C^{(1)} \varphi^{(3)}(-\mathbf{p})}^T b^{(1)}(\mathbf{p}) + \overline{\eta_C^{(2)} \varphi^{(4)}(-\mathbf{p})}^T b^{(2)}(\mathbf{p}) \right) \exp(ipx) \right. \\
&\quad \left. + \left(\overline{-\eta_C'^{(1)} \varphi^{(1)}(-\mathbf{p})}^T a^{(1)\dagger}(\mathbf{p}) + \overline{\eta_C'^{(2)} \varphi^{(2)}(-\mathbf{p})}^T a^{(2)\dagger}(\mathbf{p}) \right) \exp(-ipx) \right]
\end{aligned}$$

これが、 $\psi(x)$ で書けるためには、 $\eta_C^{(1)} = \eta_C^{(2)} = -\eta_C'^{(1)} = \eta_C'^{(2)}$ とする：

$$\begin{aligned}
&= -\eta_C \eta_\gamma^* U_C \int \frac{d\mathbf{p}}{2E} \left[\left(\overline{\varphi^{(3)}(-\mathbf{p})}^T b^{(1)}(\mathbf{p}) + \overline{\varphi^{(4)}(-\mathbf{p})}^T b^{(2)}(\mathbf{p}) \right) \exp(ipx) \right. \\
&\quad \left. + \left(\overline{\varphi^{(1)}(\mathbf{p})}^T a^{(1)\dagger}(\mathbf{p}) + \overline{\varphi^{(2)}(\mathbf{p})}^T a^{(2)\dagger}(\mathbf{p}) \right) \exp(-ipx) \right] \\
&= -\eta_C \eta_\gamma^* U_C \int \frac{d\mathbf{p}}{2E} \sum_{\lambda=1}^2 \left[\overline{\varphi^{(\lambda+2)}(-\mathbf{p})}^T b^{(\lambda)}(\mathbf{p}) \exp(ipx) + \overline{\varphi^{(\lambda)}(\mathbf{p})}^T a^{(\lambda)\dagger}(\mathbf{p}) \exp(-ipx) \right] = -\eta_C \eta_\gamma^* U_C \overline{\psi(x)}^T
\end{aligned}$$

になる。従つて・・・、

$\mathcal{U}_C a^{(\lambda)}(\mathbf{p}) \mathcal{U}_C^{-1} = \eta_C^{(\lambda)} b^{(\lambda)}(\mathbf{p}), \mathcal{U}_C b^{(\lambda)\dagger}(\mathbf{p}) \mathcal{U}_C^{-1} = \eta_C'^{(\lambda)} a^{(\lambda)\dagger}(\mathbf{p})$ with $(\eta_C = \eta_C^{(1)} = -\eta_C^{(2)} = \eta_C'^{(1)} = -\eta_C'^{(2)})$
$\mathcal{U}_C \psi(x) \mathcal{U}_C^{-1} = -\eta_C \eta_\gamma^* U_C \overline{\psi(x)}^T$ (通常、 $\eta_\gamma = -1$), $U_C = \eta_\gamma \gamma_5 \gamma^1 \gamma^3 = \eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}$, $\gamma^\mu = -U_C \gamma^{\mu T} U_C^{-1}$

になる。

第四節：CPT変換

スカラー (スピノン)

$$P: \mathcal{U}_P \phi(x) \mathcal{U}_P^{-1} \equiv \phi(t, -\mathbf{x}) = \eta_p \phi(t, \mathbf{x}), \mathcal{U}_P \phi^\dagger(x) \mathcal{U}_P^{-1} \equiv \phi^\dagger(t, -\mathbf{x}) = \eta_p^* \phi^\dagger(t, \mathbf{x})$$

$$T: \mathcal{U}_T \phi(x) \mathcal{U}_T^{-1} \equiv \phi(-t, \mathbf{x}) = \eta_T \phi^\dagger(t, \mathbf{x}), \mathcal{U}_T \phi^\dagger(x) \mathcal{U}_T^{-1} \equiv \phi^\dagger(-t, \mathbf{x}) = \eta_T^* \phi(t, \mathbf{x})$$

$$C: \mathcal{U}_C \phi(x) \mathcal{U}_C^{-1} = \eta_C \phi^\dagger(x), \mathcal{U}_C \phi^\dagger(x) \mathcal{U}_C^{-1} = \eta_C \phi(x)$$

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ベクター (スピノン 1)

$$P: \mathcal{U}_p A(x) \mathcal{U}_p^{-1} \equiv A(t, -x) = -\eta_p A(t, x), \mathcal{U}_p A_0(x) \mathcal{U}_p^{-1} \equiv A_0(t, -x) = \eta_p A_0(t, x) \quad (\eta_p = \pm 1)$$

$$T: \mathcal{U}_T A(x) \mathcal{U}_T^{-1} \equiv A(-t, x) = -\eta_T A(t, x), \mathcal{U}_T A_0(x) \mathcal{U}_T^{-1} \equiv A_0(-t, x) = \eta_T A_0(t, x) \quad (\eta_T = \pm 1)$$

$$C: \mathcal{U}_C A_\mu(x) \mathcal{U}_C^{-1} = \eta_C A_\mu(x) \quad (\eta_C = -1)$$

フェルミオン (スピノン 1 / 2)

$$P: \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} \equiv \psi(t, -x) = \eta_p \gamma^0 \psi(t, x), \mathcal{U}_T \overline{\psi(x)} \mathcal{U}_T^{-1} \equiv \overline{\psi(t, -x)} = \eta_p^* \psi^\dagger(t, x)$$

$$T: \mathcal{U}_T \psi(x) \mathcal{U}_T^{-1} \equiv \psi(-t, x) = -\eta_T \gamma^0 U_T \overline{\psi(t, x)}^T, \mathcal{U}_T \overline{\psi(x)} \mathcal{U}_T^{-1} \equiv \overline{\psi(-t, x)} = -\eta_T^* \psi^T(t, x) U_T^\dagger \gamma^0$$

$$\Rightarrow \psi(t, x) = \left[-\eta_T \overline{\psi(-t, x)} \gamma^0 U_T \right]^T \stackrel{\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}}{=} \left[-\eta_T \overline{\psi(-t, x)} U_T \gamma^{0T} \right]^T$$

$$= -\eta_T \gamma^0 U_T^T \overline{\psi(-t, x)}^T = -\eta_T \gamma^0 (-U_T) \overline{\psi(-t, x)}^T = \eta_T \gamma^0 U_T \overline{\psi(-t, x)}^T$$

$$\Rightarrow \overline{\psi(t, x)} = \left[-\eta_T^* U_T^\dagger \gamma^0 \psi(-t, x) \right]^T = -\eta_T^* [U_T^\dagger \gamma^0 \psi(-t, x)]^T$$

$$= -\eta_T^* [\gamma^{0T} U_T^\dagger \psi(-t, x)]^T = -\eta_T^* \psi^T(-t, x) (-U_T^\dagger) \gamma^0 = \eta_T^* \psi^T(-t, x) U_T^\dagger \gamma^0$$

$$\text{with } U_T = \eta_\gamma \gamma^1 \gamma^3 = \eta_\gamma \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \quad (\text{通常}, \eta_\gamma = 1), \quad U_T^* = -U_T^T = U_T, \quad \gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}$$

$$C: \mathcal{U}_C \psi(x) \mathcal{U}_C^{-1} = -\eta_C \eta_\gamma^* U_C \overline{\psi(x)}^T \quad (\text{通常}, \eta_C = -1)$$

$$\Rightarrow \mathcal{U}_C \overline{\psi(x)} \mathcal{U}_C^{-1} = (\mathcal{U}_C \psi(x) \mathcal{U}_C^{-1})^\dagger \gamma^0 = \left(-\eta_\gamma \eta_C^* U_C (\psi^\dagger(x) \gamma^0)^T \right)^\dagger \gamma^0$$

$$= -\eta_C^{(1)*} \eta_C \left(U_C \gamma^{0T} \psi(x)^{T\dagger} \right)^\dagger \gamma^0 = -\eta_C^* \eta_\gamma \psi^T(x) \gamma^{0T\dagger} U_C^\dagger \gamma^0$$

$$= -\eta_C^* \eta_\gamma \psi^T(x) \gamma^{0T} U_C^{-1} \gamma^0 = \eta_C^* \eta_\gamma \psi^T(x) U_C^{-1} \gamma^0 \gamma^0 = \eta_C^* \eta_\gamma \psi^T(x) U_C^{-1}$$

$$\text{with } U_C = \eta_\gamma \gamma_5 \gamma^1 \gamma^3 = \eta_\gamma \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \gamma^\mu = -U_C \gamma^{\mu T} U_C^{-1}, \quad U_C^{-1} = U_C^\dagger, \quad U_C^T = -U_C \quad (\text{通常}, \eta_\gamma = -1)$$

ベクター (スピノン 1) を物理自由度である \mathbf{E} と \mathbf{H} に直しておくと · · ·

$$\boxed{\mathbf{E}^i = F^{i0}} = \partial^i A^0 - \partial^0 A^i = -\nabla^i A^0 - \partial_t A^i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \nabla^i (-A^j) + \nabla^j (-A^i) = -(\nabla^i A^j - \nabla^j A^i) = -\varepsilon_{ijk} (\nabla \times \mathbf{A})^k$$

$$\mathbf{H}^i = -\frac{1}{2} \varepsilon^{ijk} F_{jk} = -\frac{1}{2} \varepsilon^{ijk} \left(-\varepsilon_{jki'} (\nabla \times \mathbf{A})^{i'} \right) = \frac{1}{2} \varepsilon^{ijk} \varepsilon_{jki'} (\nabla \times \mathbf{A})^{i'}$$

$$= \frac{1}{2} 2\delta_{i'}^i (\nabla \times \mathbf{A})^i = (\nabla \times \mathbf{A})^i \Rightarrow \boxed{F_{ij} = -\varepsilon_{ijk} \mathbf{H}^k = F^{ij}}$$

$$P: \mathbf{E}^i(t, -x) = -(-\nabla^i) \eta_p A^0(t, x) - \partial_t (-\eta_p A^i(t, x)) = -\eta_p \mathbf{E}^i(t, x)$$

$$\mathbf{H}^i(t, -x) = (\nabla \times \mathbf{A}(-t, x))^i = ((-\nabla) \times (-\eta_p \mathbf{A}(t, x)))^i = \eta_p \mathbf{H}^i(t, x)$$

$$T: \mathbf{E}^i(-t, x) = -\nabla^i (\eta_T A^0(t, x)) - (-\partial_t) (-\eta_T A^i(t, x)) = \eta_T \mathbf{E}^i(t, x)$$

$$\mathbf{H}^i(-t, x) = (\nabla \times \mathbf{A}(-t, x))^i = (\nabla \times (-\eta_T \mathbf{A}(t, x)))^i = -\eta_T \mathbf{H}^i(t, x)$$

$$C: \mathcal{U}_C \mathbf{E}^i(x) \mathcal{U}_C^{-1} = -\nabla^i \mathcal{U}_C A^0(x) \mathcal{U}_C^{-1} - \partial_t \mathcal{U}_C \mathbf{A}^i(x) \mathcal{U}_C^{-1} = \eta_C \mathbf{E}^i(x) \quad (\eta_C = -1)$$

$$\mathcal{U}_C \mathbf{H}^i(x) \mathcal{U}_C^{-1} = (\nabla \times \mathcal{U}_C \mathbf{A}(x) \mathcal{U}_C^{-1})^i = (\nabla \times (-\eta_T \mathbf{A}(t, \mathbf{x})))^i = \eta_C \mathbf{H}^i(t, \mathbf{x})$$

以上を用いて、具体的に幾つかの相互作用の変換性を調べてみる。

1) $\phi^\dagger(x)\phi(x)$

$$P: \mathcal{U}_P \phi^\dagger(t, \mathbf{x}) \phi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \eta_p \phi^\dagger(t, -\mathbf{x}) \eta_p^* \phi(t, -\mathbf{x}) = \phi^\dagger(t, -\mathbf{x}) \phi(t, -\mathbf{x}) \Rightarrow P = +$$

$$T: \mathcal{U}_T \phi^\dagger(t, \mathbf{x}) \phi(t, \mathbf{x}) \mathcal{U}_T^{-1} \stackrel{\text{順序交換}}{=} \eta_T \phi^\dagger(-t, \mathbf{x}) \eta_T^* \phi(-t, \mathbf{x}) = \phi^\dagger(-t, \mathbf{x}) \phi(-t, \mathbf{x}) \Rightarrow T = +$$

$$C: \mathcal{U}_C \phi(x) \phi^\dagger(x) \mathcal{U}_C^{-1} = \eta_C \phi^\dagger(x) \eta_C^* \phi(x) \Rightarrow C = +$$

2) $\phi^\dagger(x) \overrightarrow{\partial_\mu} \phi(x) = \phi^\dagger(x) \partial_\mu \phi(x) - \partial_\mu(\phi^\dagger(x)) \phi(x)$ (以下では、 $\varepsilon_\mu = (1, -1, -1, -1)$)

$$P: \mathcal{U}_P \phi^\dagger(t, \mathbf{x}) \overrightarrow{\partial_\mu} \phi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \eta_p \phi^\dagger(t, -\mathbf{x}) \varepsilon_\mu \overrightarrow{\partial_\mu}(\eta_p^* \phi(t, -\mathbf{x})) \quad (\mu: \text{not summed})$$

$$= \varepsilon_\mu \phi^\dagger(t, -\mathbf{x}) \overrightarrow{\partial_\mu} \phi(t, -\mathbf{x}) \Rightarrow P = \varepsilon_\mu$$

$$T: \mathcal{U}_T \phi^\dagger(t, \mathbf{x}) \overrightarrow{\partial_\mu} \phi(t, \mathbf{x}) \mathcal{U}_T^{-1} \stackrel{\text{順序交換}}{=} \eta_T \phi^\dagger(-t, \mathbf{x}) (-\varepsilon_\mu \overrightarrow{\partial_\mu})(\eta_T^* \phi(-t, \mathbf{x})) \quad (\mu: \text{not summed})$$

$$= -\varepsilon \phi^\dagger(-t, \mathbf{x}) \overrightarrow{\partial_\mu} \phi(-t, \mathbf{x}) \Rightarrow T = -\varepsilon_\mu$$

$$C: \mathcal{U}_C \phi(x) \overrightarrow{\partial_\mu} \phi^\dagger(x) \mathcal{U}_C^{-1} = (\eta_C \phi^\dagger(x)) \overrightarrow{\partial_\mu}(\eta_C^* \phi(x)) = -\phi(x) \overrightarrow{\partial_\mu} \phi^\dagger(x) \Rightarrow C = -$$

3) $\partial_\mu \phi^\dagger(x) \cdot \partial_\nu \phi(x)$ (以下では、 $\varepsilon_\mu = (1, -1, -1, -1)$)

$$P: \mathcal{U}_P \partial_\mu \phi^\dagger(t, \mathbf{x}) \cdot \partial_\nu \phi(t, \mathbf{x}) \mathcal{U}_P^{-1} \Big|_{\pm} = \varepsilon_\mu \partial_\mu \phi^\dagger(t, -\mathbf{x}) \cdot \varepsilon_\nu \partial_\nu(\phi(t, -\mathbf{x})) \Big|_{\pm} \quad (\mu, \nu: \text{not summed})$$

$$= \varepsilon_\mu \varepsilon_\nu \partial_\mu \eta_p \phi^\dagger(t, \mathbf{x}) \cdot \partial_\nu \eta_p^* \phi(t, \mathbf{x}) \Big|_{\pm} = \varepsilon_\mu \varepsilon_\nu \partial_\mu \phi^\dagger(t, \mathbf{x}) \cdot \partial_\nu \phi(t, \mathbf{x}) \Big|_{\pm} \Rightarrow P = \varepsilon_\mu \varepsilon_\nu$$

$$T: \mathcal{U}_T \partial_\mu \phi^\dagger(t, \mathbf{x}) \cdot \partial_\nu \phi(t, \mathbf{x}) \mathcal{U}_T^{-1} \Big|_{\pm} \stackrel{\text{順序交換}}{=} \varepsilon_\nu \partial_\nu \phi^\dagger(-t, \mathbf{x}) \cdot \varepsilon_\mu \partial_\mu \phi(-t, \mathbf{x}) \Big|_{\pm\pm} \quad (\mu: \text{not summed})$$

$$= \varepsilon_\mu \varepsilon_\nu (-\partial_\nu \eta_T \phi^\dagger(t, \mathbf{x})) \cdot (-\partial_\mu \eta_T^* \phi(t, \mathbf{x})) \Big| = \varepsilon_\mu \varepsilon_\nu \partial_\nu \phi^\dagger(t, \mathbf{x}) \cdot \partial_\mu \phi(t, \mathbf{x}) \Big|_{\pm} \Rightarrow T = \pm \varepsilon_\mu \varepsilon_\nu$$

$$C: \mathcal{U}_C \partial_\mu \phi^\dagger(t, \mathbf{x}) \cdot \partial_\nu \phi(t, \mathbf{x}) \mathcal{U}_C^{-1} = \partial_\mu(\eta_C^* \phi(x)) \partial_\nu(\eta_C \phi^\dagger(x)) \Big|_{\pm} = \partial_\nu \phi^\dagger(x) \partial_\mu \phi(x) \Big|_{\pm} \Rightarrow C = \pm$$

4) $\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 4\varepsilon^{ijk} F^{0i} F^{jk} = 4\varepsilon^{ijk} (-\mathbf{E}^i)(-\varepsilon^{jki'} \mathbf{H}^{i'}) = 4\varepsilon^{ijk} \varepsilon^{jki'} \mathbf{E}^i \mathbf{H}^{i'} = 4 \times 2\delta_{ii'}^i \mathbf{E}^i \mathbf{H}^{i'} = 8\mathbf{E}\mathbf{H}$

P: $\mathcal{U}_P \mathbf{E}(t, \mathbf{x}) \mathbf{H}(t, \mathbf{x}) \mathcal{U}_P^{-1} = \mathbf{E}(t, -\mathbf{x}) \mathbf{H}(t, -\mathbf{x}) = (-\eta_p \mathbf{E}(t, \mathbf{x})) \eta_p \mathbf{H}(t, \mathbf{x}) = -\mathbf{E}(t, \mathbf{x}) \mathbf{H}(t, \mathbf{x}) \Rightarrow P = -1$

T: $\mathcal{U}_T \mathbf{E}(t, \mathbf{x}) \mathbf{H}(t, \mathbf{x}) \mathcal{U}_T^{-1} \stackrel{\text{順序交換}}{=} \mathbf{H}(-t, \mathbf{x}) \mathbf{E}(-t, \mathbf{x}) = (-\eta_T \mathbf{H}^i(t, \mathbf{x})) (\eta_T \mathbf{E}(t, \mathbf{x})) = -\mathbf{H}(t, \mathbf{x}) \mathbf{E}(t, \mathbf{x}) \Rightarrow T = -1$

C: $\mathcal{U}_C \mathbf{E}(t, \mathbf{x}) \mathbf{H}(t, \mathbf{x}) \mathcal{U}_C^{-1} = (\eta_C \mathbf{E}(x)) (\eta_C \mathbf{H}(t, \mathbf{x})) = \mathbf{E}(t, \mathbf{x}) \mathbf{H}(t, \mathbf{x}) \Rightarrow C = +1$

5) $F_{\mu\nu} F^{\mu\nu} = -2F^{0i} F^{0i} + F^{ij} F^{ij} = -2\mathbf{E}\mathbf{E} + (-\varepsilon_{ijk} \mathbf{H}^k)(-\varepsilon_{ijk} \mathbf{H}^{k'}) = -2\mathbf{E}\mathbf{E} + 2\delta_{kk'} \mathbf{H}^k \mathbf{H}^{k'} = -2(\mathbf{E}\mathbf{E} - \mathbf{H}\mathbf{H})$

$$\begin{aligned}
P: & \mathcal{U}_P [E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x})] \mathcal{U}_P^{-1} = E(t, -\mathbf{x}) E(t, -\mathbf{x}) - H(t, -\mathbf{x}) H(t, -\mathbf{x}) \\
& = (-\eta_p E(t, \mathbf{x}))(-\eta_p E(t, \mathbf{x})) - \eta_p H(t, \mathbf{x}) \eta_p H(t, \mathbf{x}) = E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x}) \Rightarrow P = +1
\end{aligned}$$

$$\begin{aligned}
T: & \mathcal{U}_T [E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x})] \mathcal{U}_T^{-1} \stackrel{\text{順序交換}}{=} E(-t, \mathbf{x}) E(-t, \mathbf{x}) - H(-t, \mathbf{x}) H(-t, \mathbf{x}) \\
& = (\eta_T E(t, \mathbf{x}))(\eta_T E(t, \mathbf{x})) - (-\eta_T H^i(t, \mathbf{x}))(-\eta_T H^i(t, \mathbf{x})) = E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x}) \Rightarrow T = +1
\end{aligned}$$

$$\begin{aligned}
C: & \mathcal{U}_C (E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x})) \mathcal{U}_C^{-1} = (\eta_C E(x))(\eta_C E(x)) - (\eta_C H(t, \mathbf{x}))(\eta_C H(t, \mathbf{x})) \\
& = E(t, \mathbf{x}) E(t, \mathbf{x}) - H(t, \mathbf{x}) H(t, \mathbf{x}) \Rightarrow C = +1
\end{aligned}$$

6) $\bar{\psi} \Gamma \psi$

$$\begin{aligned}
P: & \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \Gamma \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \overline{\psi(t, -\mathbf{x})} \Gamma \psi(t, -\mathbf{x}) = \eta_p^* \overline{\psi(t, \mathbf{x})} \gamma^0 \Gamma \eta_p \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, -\mathbf{x})} \gamma^0 \Gamma \gamma^0 \psi(t, \mathbf{x}) \\
T: & \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \Gamma \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} \stackrel{\text{順序交換}}{=} \psi^T(-t, \mathbf{x}) \Gamma^T \overline{\psi(-t, \mathbf{x})}^T = \left[(\eta_T \psi^T(t, \mathbf{x}) U_T^\dagger \gamma^0) \Gamma (\gamma^0 \eta_T^* U_T \overline{\psi(t, \mathbf{x})}^T) \right]^T \\
& = \left[\psi^T(t, \mathbf{x}) U_T^\dagger \gamma^0 \Gamma \gamma^0 U_T \overline{\psi(t, \mathbf{x})}^T \right]^T \stackrel{\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}}{=} \left[\psi^T(t, \mathbf{x}) \gamma^{0T} U_T^\dagger \Gamma U_T \gamma^{0T} \overline{\psi(t, \mathbf{x})}^T \right]^T \\
& \stackrel{U_T^T = -U_T}{=} \overline{\psi(t, \mathbf{x})} \gamma^0 (-U_T) \Gamma^T (-U_T^\dagger) \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma^0 U_T \Gamma^T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) \\
C: & \mathcal{U}_C \overline{\psi(x)} \Gamma \psi(x) \mathcal{U}_C^{-1} = (\eta_C^* \eta_\gamma \psi^T(x) U_C^{-1}) \Gamma \left(-\eta_C \eta_\gamma^* U_C \overline{\psi(x)}^T \right) = -\psi^T(x) U_C^{-1} \Gamma U_C \overline{\psi(x)}^T \\
& \stackrel{\text{反可換}}{=} \overline{\psi(x)} (-U_C) \Gamma^T (-U_C^{-1}) \psi(x) = \overline{\psi(x)} U_C \Gamma^T U_C^{-1} \psi(x)
\end{aligned}$$

A) $\Gamma = \mathbf{I}$

$$\begin{aligned}
P: & \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \overline{\psi(t, -\mathbf{x})} \psi(t, -\mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \psi(t, \mathbf{x}) \Rightarrow P = +1 \\
T: & \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} = \overline{\psi(-t, \mathbf{x})} \psi(-t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma^0 U_T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \psi(t, \mathbf{x}) \Rightarrow T = +1 \\
C: & \mathcal{U}_C \overline{\psi(x)} \psi(x) \mathcal{U}_C^{-1} = \overline{\psi(x)} U_C U_C^{-1} \psi(x) = \overline{\psi(x)} \psi(x) \Rightarrow C = +1
\end{aligned}$$

B) $\Gamma = \gamma_5$

$$\begin{aligned}
P: & \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \gamma_5 \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} = \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_5 \gamma^0 \psi(t, \mathbf{x}) = -\overline{\psi(t, \mathbf{x})} \gamma_5 \psi(t, \mathbf{x}) \Rightarrow P = -1 \\
T: & \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \gamma_5 \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} = \overline{\psi(t, \mathbf{x})} \gamma^0 U_T \gamma_5^T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) = -\overline{\psi(t, \mathbf{x})} \gamma_5 \psi(t, \mathbf{x}) \Rightarrow T = -1 \\
C: & \mathcal{U}_C \overline{\psi(x)} \gamma_5 \psi(x) \mathcal{U}_C^{-1} = \overline{\psi(x)} U_C \gamma_5^T U_C^{-1} \psi(x) = \overline{\psi(x)} \gamma_5 \psi(x) \Rightarrow C = +1
\end{aligned}$$

C) $\Gamma = \gamma_\mu$ (以下では、 $\varepsilon_\mu = (1, -1, -1, -1)$)

$$\begin{aligned}
P: & \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \gamma_\mu \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} = (\eta_p \overline{\psi(t, \mathbf{x})} \gamma^0) \Gamma \eta_p^* \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_\mu \gamma^0 \psi(t, \mathbf{x}) \Rightarrow P = \varepsilon_\mu \\
T: & \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \gamma_\mu \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} = \overline{\psi(t, \mathbf{x})} \gamma^0 U_T \gamma_\mu^T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) \stackrel{\gamma^\mu = U_T \gamma^{\mu T} U_T^{-1}}{=} \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_\mu U_T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) \\
& = \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_\mu \gamma^0 \psi(t, \mathbf{x}) \Rightarrow T = \varepsilon_\mu \\
C: & \mathcal{U}_C \overline{\psi(x)} \gamma_\mu \psi(x) \mathcal{U}_C^{-1} = \overline{\psi(x)} U_C \gamma_\mu^T U_C^{-1} \psi(x) \stackrel{\gamma^\mu = -U_C \gamma^{\mu T} U_C^{-1}}{=} \overline{\psi(x)} (-\gamma_\mu) \psi(x) \Rightarrow C = -1
\end{aligned}$$

D) $\Gamma = \gamma_5 \gamma_\mu$ (以下では、 $\varepsilon_\mu = (1, -1, -1, -1)$)

$$\begin{aligned}
P: \mathcal{U}_P \overline{\psi(t, \mathbf{x})} \gamma_5 \gamma_\mu \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} &= \overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_5 \gamma_\mu \gamma^0 \psi(t, \mathbf{x}) = -\overline{\psi(t, \mathbf{x})} \gamma_5 \gamma^0 \gamma_\mu \gamma^0 \psi(t, \mathbf{x}) \Rightarrow P = -\varepsilon_\mu \\
T: \mathcal{U}_T \overline{\psi(t, \mathbf{x})} \gamma_5 \gamma_\mu \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} &= \overline{\psi(t, \mathbf{x})} \gamma^0 U_T (\gamma_5 \gamma_\mu)^T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma^0 U_T \gamma_\mu^T \gamma_5 U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) \\
&\stackrel{[U_T^\dagger, \gamma_5]=0}{=} -\overline{\psi(t, \mathbf{x})} \gamma^0 U_T \gamma_\mu^T U_T^\dagger \gamma^0 \gamma_5 \psi(t, \mathbf{x}) = -\overline{\psi(t, \mathbf{x})} \gamma^0 \gamma_\mu \gamma^0 \gamma_5 \psi(t, \mathbf{x}) = \overline{\psi(t, \mathbf{x})} \gamma_5 \gamma^0 \gamma_\mu \gamma^0 \psi(t, \mathbf{x}) \Rightarrow T = \varepsilon_\mu \\
C: \mathcal{U}_C \overline{\psi(x)} \gamma_5 \gamma_\mu \psi(x) \mathcal{U}_C^{-1} &= \overline{\psi(x)} U_C (\gamma_5 \gamma_\mu)^T U_C^{-1} \psi(x) = \overline{\psi(x)} \gamma_5 \gamma_\mu^T \gamma_5 U_C^{-1} \psi(x) \\
&\stackrel{[U_C^{-1}, \gamma_5]=0}{=} \overline{\psi(x)} U_C \gamma_\mu^T U_C^{-1} \gamma_5 \psi(x) = -\overline{\psi(x)} \gamma_\mu \gamma_5 \psi(x) = \overline{\psi(x)} \gamma_5 \gamma_\mu \psi(x) \Rightarrow T = +1
\end{aligned}$$

E) $T=[\gamma_\mu, \gamma_\nu]$ (以下では、 $\varepsilon_{\mu\nu}=(1, -1, -1, -1)$)

$$\begin{aligned}
P: \mathcal{U}_P \overline{\psi(t, \mathbf{x})} [\gamma^\mu, \gamma^\nu] \psi(t, \mathbf{x}) \mathcal{U}_P^{-1} &= \overline{\psi(t, \mathbf{x})} \gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0 \psi(t, \mathbf{x}) \Rightarrow P = \varepsilon_\mu \varepsilon_\nu \\
T: \mathcal{U}_T \overline{\psi(t, \mathbf{x})} [\gamma^\mu, \gamma^\nu] \psi(t, \mathbf{x}) \mathcal{U}_T^{-1} &= \overline{\psi(t, \mathbf{x})} \gamma^0 U_T [\gamma^\mu, \gamma^\nu]^T U_T^\dagger \gamma^0 \psi(t, \mathbf{x}) \\
&\stackrel{(*) \gamma^0 U_T [\gamma^\mu, \gamma^\nu]^T U_T^\dagger \gamma^0 = -\gamma^0 U_T [\gamma^{\mu T}, \gamma^{\nu T}] U_T^\dagger \gamma^0 \stackrel{\gamma^{\mu T}=U_T \gamma^{\mu T} U_T^{-1}}{=} -\gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0}{=} -\overline{\psi(t, \mathbf{x})} (-\gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0) \psi(t, \mathbf{x}) = -\overline{\psi(t, \mathbf{x})} \gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0 \psi(t, \mathbf{x}) \Rightarrow P = -\varepsilon_\mu \varepsilon_\nu \\
C: \mathcal{U}_C \overline{\psi(x)} [\gamma^\mu, \gamma^\nu] \psi(x) \mathcal{U}_C^{-1} &= \overline{\psi(x)} U_C [\gamma^\mu, \gamma^\nu]^T U_C^{-1} \psi(x) = -\overline{\psi(x)} U_C [\gamma^{\mu T}, \gamma^{\nu T}] U_C^{-1} \psi(x) \\
&\stackrel{\gamma^{\mu T}=-U_C \gamma^{\mu T} U_C^{-1}}{=} -\overline{\psi(x)} [\gamma^\mu, \gamma^\nu] \psi(x) \Rightarrow T = -1
\end{aligned}$$

以上から、 $\varepsilon_{\mu\nu}=(1, -1, -1, -1)$

	種類	P	T	C	CPT
$\phi^\dagger \phi$	スカラ一	+1	+1	+1	+1
$E \cdot H$	“	-1	-1	+1	+1
$E^2 - H^2$	“	+1	+1	+1	+1
$\bar{\psi} \psi$	“	+1	+1	+1	+1
$\bar{\psi} \gamma_5 \psi$	“	-1	-1	+1	+1
$\phi^\dagger \overleftrightarrow{\partial}_\mu \phi$	ベクトル	ε_μ	ε_μ	-1	-1
$\bar{\psi} \gamma_\mu \psi$	“	ε_μ	ε_μ	-1	-1
$\bar{\psi} \gamma_5 \gamma_\mu \psi$	“	$-\varepsilon_\mu$	ε_μ	+1	-1
$\partial_\mu \phi^\dagger(x) \cdot \partial_\nu \phi(x) \Big _{\pm}$	テンソル	$\varepsilon_\mu \varepsilon_\nu$	$\pm \varepsilon_\mu \varepsilon_\nu$	± 1	+1
$\bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$	“	$\varepsilon_\mu \varepsilon_\nu$	$-\varepsilon_\mu \varepsilon_\nu$	-1	+1

がわかる。これより、CPT の値は

$$\text{スカラー} = +1, \text{ベクトル} = -1, \text{テンソル} = +1$$

がわかり、例えば、これから作られるローレンツスカラーは、「スカラー = +1、スカラー × ベクトル × ベクトル = +1、テンソル × ベクトル × ベクトル = +1」とすべて

$$\text{CPT} = +1 \text{ を示す}$$

ことになる。従って、思いつく場合はすべて

$$\text{CPT} \text{ は不変}$$

である。

第五節：C P の破れ

湯川相互作用

$$\mathcal{L} = f_{ab}^c \overline{\psi}_a \psi_b \phi_c + (\text{h.c.})$$

において C P 変換は・・・

$$C: \mathcal{U}_C \phi(x) \mathcal{U}_C^\dagger = \eta_C \phi^\dagger(x)$$

$$\begin{aligned} C: \mathcal{U}_C \overline{\psi}_a(x) \Gamma \psi_b(x) \mathcal{U}_C^{-1} &= (\eta_{ac}^* \eta_{a\gamma} \psi_a^T(x) U_C^{-1}) \Gamma \left(-\eta_{bc} U_C \overline{\psi}_b(x)^T \right) = \eta_{ac}^* \eta_{a\gamma} \eta_{bc} \eta_{b\gamma}^* \overline{\psi}_b(x) U_C \Gamma^T U_C^{-1} \psi_a(x) \\ &\equiv \eta_a^* \eta_b \overline{\psi}_b(x) U_C \Gamma^T U_C^{-1} \psi_a(x) \quad (\eta_b \equiv \eta_{bc} \eta_{b\gamma}^*) \end{aligned}$$

に注意して、

$$P: \mathcal{U}_P \phi(x) \mathcal{U}_P^{-1} = \eta_P \phi(t, x), \quad C: \mathcal{U}_C \phi(x) \mathcal{U}_C^\dagger = \eta_C \phi^\dagger(x)$$

$$CP: \mathcal{U}_C \mathcal{U}_P \phi(x) \mathcal{U}_P^{-1} \mathcal{U}_C^\dagger = \mathcal{U}_C \eta_P \phi(x) \mathcal{U}_C^\dagger = \eta_P \eta_C \phi^\dagger(x)$$

$$P: \mathcal{U}_P \overline{\psi}_a(t, x) \psi_b(t, x) \mathcal{U}_P^{-1} = \overline{\psi}_a(t, -x) \psi_b(t, -x) = \overline{\psi}_a(t, x) \gamma^0 \gamma^0 \psi_b(t, x) = \overline{\psi}_a(t, x) \psi_b(t, x)$$

$$C: \mathcal{U}_C \overline{\psi}_a(x) \psi_b(x) \mathcal{U}_C^{-1} = \eta_a^* \eta_b \overline{\psi}_b(x) \psi_a(x)$$

$$CP: \mathcal{U}_C \mathcal{U}_P \left(\overline{\psi}_a(x) \psi_b(x) \right) \mathcal{U}_P^{-1} \mathcal{U}_C^{-1} = \mathcal{U}_C \left(\overline{\psi}_a(x) \psi_b(x) \right) \mathcal{U}_C^{-1} = \eta_a^* \eta_b \overline{\psi}_b(x) \psi_a(x)$$

なので、

$$\begin{aligned} \mathcal{U}_C \mathcal{U}_P \mathcal{L} \mathcal{U}_P^{-1} \mathcal{U}_C^\dagger &= \mathcal{U}_C \mathcal{U}_P f_{ab}^c \overline{\psi}_a \psi_b \phi_c \mathcal{U}_P^{-1} \mathcal{U}_C^\dagger + (\text{h.c.}) = f_{ab}^c \eta_a^* \eta_b \eta_p \eta_C \overline{\psi}_b \psi_a \phi_c^\dagger + (\text{h.c.}) \\ &= f_{ab}^c \eta_a^* \eta_b \eta_p \eta_C \overline{\psi}_b \psi_a \phi_c^\dagger + f_{ab}^{c*} \eta_a \eta_b^* \eta_p^* \eta_C^* \overline{\psi}_a \psi_b \phi_c^\dagger \end{aligned}$$

になる。もし C P が保存すれば

$$\mathcal{U}_C \mathcal{U}_P \mathcal{L} \mathcal{U}_P^{-1} \mathcal{U}_C^\dagger = \mathcal{L}$$

より

$$f_{ab}^c \eta_a^* \eta_b \eta_p \eta_C \overline{\psi}_b \psi_a \phi_c^\dagger + f_{ab}^{c*} \eta_a \eta_b^* \eta_p^* \eta_C^* \overline{\psi}_a \psi_b \phi_c = f_{ab}^c \overline{\psi}_a \psi_b \phi_c + f_{ab}^{c*} \overline{\psi}_b \psi_a \phi_c^\dagger$$

の条件を得る。従って、

$$f_{ab}^c \eta_a^* \eta_b \eta_p \eta_C = f_{ab}^{c*}$$

が条件になる。ところで、場の位相は自由に取れるので、 $\eta_a^*, \eta_b, \eta_p, \eta_c$ を適当にとれば、常に

$$\eta_a^* \eta_b \eta_p \eta_c = \text{実数}$$

にすることができる。つまり、

$$f_{ab}^c = f_{ab}^{c*}$$

になる。従って、

湯川結合がすべて実数なら C P は保存する

ことがわかる。同様に、(ヒッグス) スカラーの相互作用でも同じ結論である。

Appendix 1 : スピン1の状態

スピン1の演算子は

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, S^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } \begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \mathbf{a}^{(3)} \end{pmatrix}$$

であり、進行方向の成分を

$$S_p \equiv \frac{\mathbf{p}}{|\mathbf{p}|} S$$

で表すと

$$\begin{aligned} S_p \equiv \frac{\mathbf{p}}{|\mathbf{p}|} S &= \frac{\mathbf{p}^1 S^1 + \mathbf{p}^2 S^2 + \mathbf{p}^3 S^3}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & -i\mathbf{p}^3 & i\mathbf{p}^2 \\ i\mathbf{p}^3 & 0 & -i\mathbf{p}^1 \\ -i\mathbf{p}^2 & i\mathbf{p}^1 & 0 \end{pmatrix} \\ S_p \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & -i\mathbf{p}^3 & i\mathbf{p}^2 \\ i\mathbf{p}^3 & 0 & -i\mathbf{p}^1 \\ -i\mathbf{p}^2 & i\mathbf{p}^1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = S \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} -i\mathbf{p}^3 y + i\mathbf{p}^2 z = |\mathbf{p}| S x = s x \\ i\mathbf{p}^3 x - i\mathbf{p}^1 z = |\mathbf{p}| S y = s y \\ -i\mathbf{p}^2 x + i\mathbf{p}^1 y = |\mathbf{p}| S z = s z \end{cases} \quad (s = |\mathbf{p}| S) \\ \Rightarrow \begin{cases} -i\mathbf{p}^3 y s + i\mathbf{p}^2 (-i\mathbf{p}^2 x + i\mathbf{p}^1 y) = s^2 x \\ i\mathbf{p}^3 x s - i\mathbf{p}^1 (-i\mathbf{p}^2 x + i\mathbf{p}^1 y) = s^2 y \end{cases} &\Rightarrow \begin{cases} (-i\mathbf{p}^3 s + i\mathbf{p}^2 i\mathbf{p}^1) y = (|\mathbf{p}| s^2 + i\mathbf{p}^2 i\mathbf{p}^2) x \\ (i\mathbf{p}^3 s + i\mathbf{p}^1 i\mathbf{p}^2) x = (s^2 + i\mathbf{p}^1 i\mathbf{p}^1) y \end{cases} \\ \Rightarrow \frac{-i\mathbf{p}^3 s + i\mathbf{p}^2 i\mathbf{p}^1}{s^2 + i\mathbf{p}^1 i\mathbf{p}^1} = \frac{s^2 + i\mathbf{p}^2 i\mathbf{p}^2}{i\mathbf{p}^3 s + i\mathbf{p}^1 i\mathbf{p}^2} &\Rightarrow \frac{-i\mathbf{p}^3 s - \mathbf{p}^1 \mathbf{p}^2}{s^2 - \mathbf{p}^1 \mathbf{p}^1} = \frac{s^2 - \mathbf{p}^2 \mathbf{p}^2}{i\mathbf{p}^3 s - \mathbf{p}^1 \mathbf{p}^2} \Rightarrow s^2 (s^2 - |\mathbf{p}|^2) = 0 \\ \Rightarrow s = 0, s = \pm |\mathbf{p}| &\Rightarrow S = 0, S = \pm 1 \end{aligned}$$

より、固有値は期待通り、 $S = 0, S = \pm 1$ になる。対応する固有状態は・・・

$S=0$ (縦波成分で運動量に平行)

$$S_p \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & -i\mathbf{p}^3 & i\mathbf{p}^2 \\ i\mathbf{p}^3 & 0 & -i\mathbf{p}^1 \\ -i\mathbf{p}^2 & i\mathbf{p}^1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} -i\mathbf{p}^3 y + i\mathbf{p}^2 z = 0 \\ i\mathbf{p}^3 x - i\mathbf{p}^1 z = 0 \\ -i\mathbf{p}^2 x + i\mathbf{p}^1 y = 0 \end{cases} \Rightarrow \begin{cases} x = \mathbf{p}^1 \\ y = \mathbf{p}^2 \\ z = \mathbf{p}^3 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{p}$$

$S=\pm 1$ (横波成分で運動量に垂直: 物理自由度としての 2 成分を与える)

$$\begin{aligned} \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & -i\mathbf{p}^3 & i\mathbf{p}^2 \\ i\mathbf{p}^3 & 0 & -i\mathbf{p}^1 \\ -i\mathbf{p}^2 & i\mathbf{p}^1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = S \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} -i\mathbf{p}^3 y + i\mathbf{p}^2 z = S|\mathbf{p}|x \\ i\mathbf{p}^3 x - i\mathbf{p}^1 z = S|\mathbf{p}|y \\ -i\mathbf{p}^2 x + i\mathbf{p}^1 y = S|\mathbf{p}|z \end{cases} (S=\pm 1) \\ \Rightarrow \begin{cases} -i\mathbf{p}^3 y S|\mathbf{p}| + i\mathbf{p}^2 (-i\mathbf{p}^2 x + i\mathbf{p}^1 y) = S^2 |\mathbf{p}|^2 x = |\mathbf{p}|^2 x \\ i\mathbf{p}^3 x S|\mathbf{p}| - i\mathbf{p}^1 (-i\mathbf{p}^2 x + i\mathbf{p}^1 y) = S^2 |\mathbf{p}|^2 y = |\mathbf{p}|^2 y \end{cases} \Rightarrow \begin{cases} (-i\mathbf{p}^3 S|\mathbf{p}| + i\mathbf{p}^2 i\mathbf{p}^1) y = (|\mathbf{p}|^2 + i\mathbf{p}^2 i\mathbf{p}^2) x \\ (i\mathbf{p}^3 S|\mathbf{p}| + i\mathbf{p}^1 i\mathbf{p}^2) x = (|\mathbf{p}|^2 + i\mathbf{p}^1 i\mathbf{p}^1) y \end{cases} \\ \Rightarrow y = \frac{i\mathbf{p}^3 S|\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^2}{|\mathbf{p}|^2 - \mathbf{p}^1 \mathbf{p}^1} x = \frac{i\mathbf{p}^3 S|\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^2}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} x \Rightarrow z = \frac{-i\mathbf{p}^2 x + i\mathbf{p}^1 y}{S|\mathbf{p}|} = \frac{i}{S|\mathbf{p}|} (\mathbf{p}^1 y - \mathbf{p}^2 x) \\ z = \frac{i}{S|\mathbf{p}|} \left(\mathbf{p}^1 \frac{i\mathbf{p}^3 S|\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^2}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} - \mathbf{p}^2 \right) x = \frac{i}{S|\mathbf{p}|} \frac{i\mathbf{p}^1 \mathbf{p}^3 S|\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^1 \mathbf{p}^2 - \mathbf{p}^2 (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3)}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} x \\ = \frac{i}{S|\mathbf{p}|} \frac{i\mathbf{p}^1 \mathbf{p}^3 S|\mathbf{p}| - \mathbf{p}^2 |\mathbf{p}|^2}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} x = \frac{i}{S} \frac{i\mathbf{p}^1 \mathbf{p}^3 S - \mathbf{p}^2 |\mathbf{p}|}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} x = -\frac{1}{S} \frac{\mathbf{p}^1 \mathbf{p}^3 S + i\mathbf{p}^2 |\mathbf{p}|}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} x \\ (*) (\mathbf{p}^1 \mathbf{p}^2 + i\mathbf{p}^3 S|\mathbf{p}|) (\mathbf{p}^1 \mathbf{p}^2 - i\mathbf{p}^3 S|\mathbf{p}|) = (|\mathbf{p}|^2 - \mathbf{p}^1 \mathbf{p}^1) (|\mathbf{p}|^2 - \mathbf{p}^2 \mathbf{p}^2) \end{aligned}$$

以上から、規格化した状態を $\mathbf{e}^{(\pm, //)}(\mathbf{p})$ で表すと $\cdots S^3 \mathbf{e}^{(\pm)}(\mathbf{p}) = \pm \mathbf{e}^{(\pm)}(\mathbf{p})$, $S^3 \mathbf{e}^{(//)}(\mathbf{p}) = 0$

$$\mathbf{e}^{(+)}(\mathbf{p}) = \frac{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3}{\sqrt{N}} \begin{pmatrix} 1 \\ \frac{i\mathbf{p}^3 |\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^2}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} \\ -\frac{\mathbf{p}^1 \mathbf{p}^3 + i\mathbf{p}^2 |\mathbf{p}|}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} \end{pmatrix}, \quad \mathbf{e}^{(-)}(\mathbf{p}) = \frac{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3}{\sqrt{N}} \begin{pmatrix} 1 \\ -\frac{i\mathbf{p}^3 |\mathbf{p}| + \mathbf{p}^1 \mathbf{p}^2}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} \\ \frac{-\mathbf{p}^1 \mathbf{p}^3 + i\mathbf{p}^2 |\mathbf{p}|}{\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3} \end{pmatrix}, \quad \mathbf{e}^{(//)}(\mathbf{p}) = \frac{\mathbf{p}}{|\mathbf{p}|}$$

ここに、物理自由度は、 $\mathbf{e}^{(\pm)}(\mathbf{p})$ であり、 $\mathbf{p} \cdot \mathbf{e}^{(\pm)}(\mathbf{p}) = 0$ を自動的に満たす。規格化因子 N は

$$\begin{aligned} N &= |\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3|^2 + |i\mathbf{p}^3 |\mathbf{p}| - \mathbf{p}^1 \mathbf{p}^2|^2 + |\mathbf{p}^1 \mathbf{p}^3 + i\mathbf{p}^2 |\mathbf{p}||^2 \\ &= (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3)^2 + \mathbf{p}^1 \mathbf{p}^2 \mathbf{p}^1 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3 |\mathbf{p}|^2 + \mathbf{p}^1 \mathbf{p}^3 \mathbf{p}^1 \mathbf{p}^3 + \mathbf{p}^2 \mathbf{p}^2 |\mathbf{p}|^2 \\ &= (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3)^2 + (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) \mathbf{p}^1 \mathbf{p}^1 + (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) |\mathbf{p}|^2 \\ &= (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3)^2 + (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) (\mathbf{p}^1 \mathbf{p}^1 + |\mathbf{p}|^2) \\ &= (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3 + \mathbf{p}^1 \mathbf{p}^1 + |\mathbf{p}|^2) = (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) (|\mathbf{p}|^2 + |\mathbf{p}|^2) = 2 (\mathbf{p}^2 \mathbf{p}^2 + \mathbf{p}^3 \mathbf{p}^3) |\mathbf{p}|^2 \end{aligned}$$

なので、

$$\boldsymbol{e}^{(+)}(\boldsymbol{p}) = \sqrt{\frac{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3}{2|\boldsymbol{p}|^2}} \begin{pmatrix} 1 \\ -\frac{\boldsymbol{p}^1 \boldsymbol{p}^2 - i\boldsymbol{p}^3 |\boldsymbol{p}|}{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3} \\ -\frac{\boldsymbol{p}^1 \boldsymbol{p}^3 + i\boldsymbol{p}^2 |\boldsymbol{p}|}{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3} \end{pmatrix}, \quad \boldsymbol{e}^{(-)}(\boldsymbol{p}) = \sqrt{\frac{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3}{2|\boldsymbol{p}|^2}} \begin{pmatrix} 1 \\ -\frac{\boldsymbol{p}^1 \boldsymbol{p}^2 + i\boldsymbol{p}^3 |\boldsymbol{p}|}{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3} \\ -\frac{\boldsymbol{p}^1 \boldsymbol{p}^3 - i\boldsymbol{p}^2 |\boldsymbol{p}|}{\boldsymbol{p}^2 \boldsymbol{p}^2 + \boldsymbol{p}^3 \boldsymbol{p}^3} \end{pmatrix}, \quad \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$$

になる。ここで、進行方向が 3 軸方向になる座標系に移れば、

$$\boldsymbol{e}^{(+)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i\sigma \\ 0 \end{pmatrix}, \quad \boldsymbol{e}^{(-)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i\sigma \\ 0 \end{pmatrix}, \quad \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) = \sigma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \left(\sigma = \frac{\boldsymbol{p}^3}{|\boldsymbol{p}^3|} = \pm 1 \right)$$

もとで $\boldsymbol{e}^{(1,2,3,4)}(\boldsymbol{p})$ で表すと · · ·

$$\begin{aligned} \boldsymbol{e}^{(+)}(\boldsymbol{p}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i\sigma \\ 0 \end{pmatrix} = \frac{\boldsymbol{e}^{(1)}(\boldsymbol{p}) + i\sigma \boldsymbol{e}^{(2)}(\boldsymbol{p})}{\sqrt{2}}, \quad \boldsymbol{e}^{(-)}(\boldsymbol{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i\sigma \\ 0 \end{pmatrix} = \frac{\boldsymbol{e}^{(1)}(\boldsymbol{p}) - i\sigma \boldsymbol{e}^{(2)}(\boldsymbol{p})}{\sqrt{2}}, \quad \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) = \sigma \boldsymbol{e}^{(3)}(\boldsymbol{p}) \\ \Leftrightarrow \boldsymbol{e}^{(1)}(\boldsymbol{p}) &= \frac{\boldsymbol{e}^{(+)}(\boldsymbol{p}) + \boldsymbol{e}^{(-)}(\boldsymbol{p})}{\sqrt{2}}, \quad \boldsymbol{e}^{(2)}(\boldsymbol{p}) = \frac{\boldsymbol{e}^{(+)}(\boldsymbol{p}) - \boldsymbol{e}^{(-)}(\boldsymbol{p})}{\sqrt{2}i\sigma}, \quad \boldsymbol{e}^{(3)}(\boldsymbol{p}) = \sigma \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) \\ \boldsymbol{a}(\boldsymbol{p}) &= \sum_{\lambda=1}^3 \boldsymbol{e}^{(\lambda)}(\boldsymbol{p}) \boldsymbol{a}^{(\lambda)}(\boldsymbol{p}) = \frac{\boldsymbol{e}^{(+)}(\boldsymbol{p}) + \boldsymbol{e}^{(-)}(\boldsymbol{p})}{\sqrt{2}} \boldsymbol{a}^{(1)}(\boldsymbol{p}) + \frac{\boldsymbol{e}^{(+)}(\boldsymbol{p}) - \boldsymbol{e}^{(-)}(\boldsymbol{p})}{\sqrt{2}i\sigma} \boldsymbol{a}^{(2)}(\boldsymbol{p}) + \sigma \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) \boldsymbol{a}^{(3)}(\boldsymbol{p}) \\ &= \boldsymbol{e}^{(+)}(\boldsymbol{p}) \frac{\boldsymbol{a}^{(1)}(\boldsymbol{p}) - i\sigma \boldsymbol{a}^{(2)}(\boldsymbol{p})}{\sqrt{2}} + \boldsymbol{e}^{(-)}(\boldsymbol{p}) \frac{\boldsymbol{a}^{(1)}(\boldsymbol{p}) + i\sigma \boldsymbol{a}^{(2)}(\boldsymbol{p})}{\sqrt{2}} + \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) \sigma \boldsymbol{a}^{(3)}(\boldsymbol{p}) \\ &= \boldsymbol{e}^{(+)}(\boldsymbol{p}) \boldsymbol{a}^{(-)}(\boldsymbol{p}) + \boldsymbol{e}^{(-)}(\boldsymbol{p}) \boldsymbol{a}^{(+)}(\boldsymbol{p}) + \boldsymbol{e}^{(\text{II})}(\boldsymbol{p}) \boldsymbol{a}^{(\text{II})}(\boldsymbol{p}) \end{aligned}$$

ここで

$$\boldsymbol{a}^{(+)}(\boldsymbol{p}) \equiv \frac{\boldsymbol{a}^{(1)}(\boldsymbol{p}) + i\sigma \boldsymbol{a}^{(2)}(\boldsymbol{p})}{\sqrt{2}}, \quad \boldsymbol{a}^{(-)}(\boldsymbol{p}) \equiv \frac{\boldsymbol{a}^{(1)}(\boldsymbol{p}) - i\sigma \boldsymbol{a}^{(2)}(\boldsymbol{p})}{\sqrt{2}}, \quad \boldsymbol{a}^{(\text{II})}(\boldsymbol{p}) = \sigma \boldsymbol{a}^{(3)}(\boldsymbol{p})$$

である。

Appendix 2 : ディラック方程式の解

$$(p^\mu \gamma_\mu - m) \varphi^{(\lambda)}(\boldsymbol{p}) = 0 \Rightarrow \left[E \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} - \boldsymbol{p}^i \begin{pmatrix} 0 & \boldsymbol{\sigma}^i \\ -\boldsymbol{\sigma}^i & 0 \end{pmatrix} - m \right] \begin{pmatrix} z \\ w \end{pmatrix} = 0$$

$$\begin{pmatrix} E - m & -\boldsymbol{p}^i \boldsymbol{\sigma}^i \\ \boldsymbol{p}^i \boldsymbol{\sigma}^i & -(E + m) \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = 0 \Rightarrow \begin{cases} (E - m)z = \boldsymbol{p}^i \boldsymbol{\sigma}^i w \\ (E + m)w = \boldsymbol{p}^i \boldsymbol{\sigma}^i z \end{cases} \Rightarrow (E - m)z = \boldsymbol{p}^i \boldsymbol{\sigma}^i \frac{\boldsymbol{p}^j \boldsymbol{\sigma}^j}{E + m} z \Rightarrow E^2 - m^2 = |\boldsymbol{p}|^2$$

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従って、スピノールは (w の代わりに z と記すことに) して)

$$\varphi^{(\lambda)}(\mathbf{p}) = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^i \boldsymbol{\sigma}^i}{E+m} \end{pmatrix} z, \quad \begin{pmatrix} \frac{\mathbf{p}^i \boldsymbol{\sigma}^i}{E-m} \\ 1 \end{pmatrix} z = \begin{pmatrix} \frac{\mathbf{p}^i \boldsymbol{\sigma}^i}{E-m} \\ 1 \end{pmatrix} w \quad \text{with } E = \pm \sqrt{|\mathbf{p}|^2 + m^2}$$

と表せる。前と同様、進行方向の成分を S_p で表すと、

$$\begin{aligned} S_p \equiv \frac{\mathbf{p}}{|\mathbf{p}|} \boldsymbol{\sigma} &= \frac{1}{|\mathbf{p}|} \begin{pmatrix} \mathbf{p}^3 & \mathbf{p}^1 - i\mathbf{p}^2 \\ \mathbf{p}^1 + i\mathbf{p}^2 & -\mathbf{p}^3 \end{pmatrix} \Rightarrow \varphi^{(\lambda)}(\mathbf{p}) = \begin{pmatrix} 1 \\ \frac{S_p |\mathbf{p}|}{E+m} \end{pmatrix} z, \quad \begin{pmatrix} \frac{S_p |\mathbf{p}|}{E-m} \\ 1 \end{pmatrix} z \\ S_p z &= \frac{1}{|\mathbf{p}|} \begin{pmatrix} \mathbf{p}^3 & \mathbf{p}^1 - i\mathbf{p}^2 \\ \mathbf{p}^1 + i\mathbf{p}^2 & -\mathbf{p}^3 \end{pmatrix} z = Sz \Rightarrow \begin{pmatrix} \mathbf{p}^3 & \mathbf{p}^1 - i\mathbf{p}^2 \\ \mathbf{p}^1 + i\mathbf{p}^2 & -\mathbf{p}^3 \end{pmatrix} z = |\mathbf{p}| Sz = sz \quad (s = |\mathbf{p}| S) \\ &\Rightarrow \begin{pmatrix} \mathbf{p}^3 & \mathbf{p}^1 - i\mathbf{p}^2 \\ \mathbf{p}^1 + i\mathbf{p}^2 & -\mathbf{p}^3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = s \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \mathbf{p}^3 x + (\mathbf{p}^1 - i\mathbf{p}^2) y = sx \\ (\mathbf{p}^1 + i\mathbf{p}^2) x - \mathbf{p}^3 y = sy \end{cases} \\ &\Rightarrow \begin{cases} (\mathbf{p}^1 - i\mathbf{p}^2) y = (s - \mathbf{p}^3) x \\ (\mathbf{p}^1 + i\mathbf{p}^2) x = (s + \mathbf{p}^3) y \end{cases} \Rightarrow \frac{s - \mathbf{p}^3}{\mathbf{p}^1 + i\mathbf{p}^2} = \frac{\mathbf{p}^1 - i\mathbf{p}^2}{s + \mathbf{p}^3} \Rightarrow |\mathbf{p}^1 + i\mathbf{p}^2|^2 = s^2 - \mathbf{p}^3 \mathbf{p}^3 \\ &\Rightarrow s^2 = |\mathbf{p}|^2 \Rightarrow S^2 |\mathbf{p}|^2 = |\mathbf{p}|^2 \Rightarrow S = \pm 1 \end{aligned}$$

そこで、

$$\begin{cases} (\mathbf{p}^1 - i\mathbf{p}^2) y = (S |\mathbf{p}| - \mathbf{p}^3) x \\ (\mathbf{p}^1 + i\mathbf{p}^2) x = (S |\mathbf{p}| + \mathbf{p}^3) y \end{cases} \Rightarrow \begin{cases} x = \frac{\mathbf{p}^1 - i\mathbf{p}^2}{S |\mathbf{p}| - \mathbf{p}^3} y \\ y = \frac{\mathbf{p}^1 + i\mathbf{p}^2}{S |\mathbf{p}| + \mathbf{p}^3} x \end{cases} \Rightarrow \begin{cases} z^+ = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix} y \quad \text{for } S = 1 \\ z^- = \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} x = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} y \quad \text{for } S = -1 \end{cases}$$

これらより (x, y は規格化因子で運動量に依存していることに注意)

$$\begin{aligned} \varphi^{(1) \text{ with } E>0, S=1}(\mathbf{p}) &= \begin{pmatrix} 1 \\ \frac{S_p |\mathbf{p}|}{E+m} \end{pmatrix} z^+ = \begin{pmatrix} 1 \\ \frac{|\mathbf{p}|}{|E|+m} \end{pmatrix} z^+ = \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ \frac{|\mathbf{p}|}{|E|+m} \left(\frac{1}{\mathbf{p}^1 + i\mathbf{p}^2} \right) \end{pmatrix} x = \begin{pmatrix} \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \\ \frac{|\mathbf{p}|}{|E|+m} \left(\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \right) \end{pmatrix} y \\ \varphi^{(2) \text{ with } E>0, S=-1}(\mathbf{p}) &= \begin{pmatrix} 1 \\ \frac{S_p |\mathbf{p}|}{|E|+m} \end{pmatrix} z^- = \begin{pmatrix} 1 \\ -\frac{|\mathbf{p}|}{|E|+m} \end{pmatrix} z^- = \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \\ -\frac{|\mathbf{p}|}{|E|+m} \left(-\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \right) \end{pmatrix} y = \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ -\frac{|\mathbf{p}|}{|E|+m} \left(-\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \right) \end{pmatrix} x \end{aligned}$$

$$\varphi^{(3) \text{ with } E < 0, S=1}(\mathbf{p}) = \begin{pmatrix} S_p |\mathbf{p}| \\ E - m \\ 1 \end{pmatrix} z^+ = \begin{pmatrix} |\mathbf{p}| \\ -|E| - m \\ 1 \end{pmatrix} z^+ = \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} \\ 1 \\ \frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \end{pmatrix} x = \begin{pmatrix} -\frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} \mathbf{p}^1 - i\mathbf{p}^2 \\ \frac{|\mathbf{p}| - \mathbf{p}^3}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \\ \frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \\ 1 \end{pmatrix} y$$

$$\varphi^{(4), E < 0, S=-1}(\mathbf{p}) = \begin{pmatrix} S_p |\mathbf{p}| \\ E - m \\ 1 \end{pmatrix} z^- = \begin{pmatrix} |\mathbf{p}| \\ |E| + m \\ 1 \end{pmatrix} z^- = \begin{pmatrix} \frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} \\ -\frac{\mathbf{p}^1 - i\mathbf{p}^2}{|\mathbf{p}| + \mathbf{p}^3} \\ 1 \end{pmatrix} y = \begin{pmatrix} \frac{|\mathbf{p}|}{|E| + m} \begin{pmatrix} 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} \\ 1 \\ -\frac{\mathbf{p}^1 + i\mathbf{p}^2}{|\mathbf{p}| - \mathbf{p}^3} \end{pmatrix} x$$

また、

$$\varphi^{(\lambda)\dagger} \varphi^{(\lambda')} = I$$

なので、 $u_\alpha^{(\lambda)}(\mathbf{p})$ のかわりに $\varphi_\alpha^{(\lambda)}(\mathbf{p})$ に置き換える。

$$a_\alpha(\mathbf{p}) = \sum_{\lambda=1}^4 u_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p}) \Rightarrow a_\alpha(\mathbf{p}) = \sum_{\lambda=1}^4 \varphi_\alpha^{(\lambda)}(\mathbf{p}) a^{(\lambda)}(\mathbf{p})$$

にできる。