

付録 E グリーン展開 (作成中)

p 個のグリーン成分

オーダー p のパラボーズ・パラフェルミ粒子の生成・消滅演算子 \hat{a}_k と \hat{a}_k^\dagger ($k=1, \dots, f$) を

- p 個の生成・消滅演算子 (グリーン成分) : $\hat{a}_k^{(\alpha)}, \hat{a}_k^{(\alpha)\dagger}$ ($\alpha=1, \dots, p$)

を用いて表わす。 \hat{a}_k と \hat{a}_k^\dagger は、

$$\hat{a}_k = \sum_{\alpha=1}^p \hat{a}_k^{(\alpha)}, \quad \hat{a}_k^\dagger = \sum_{\alpha=1}^p \hat{a}_k^{(\alpha)\dagger} \quad (1.1)$$

で定義される。その際、パラフェルミ粒子では同じ成分はフェルミ粒子の反交換関係 :

$\{\hat{a}, \hat{a}^\dagger\} = 1$ 、 $\{\hat{a}, \hat{a}\} = 0$ 、 $\{\hat{a}^\dagger, \hat{a}^\dagger\} = 0$ を満たすとする。つまり、

$$\{\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}\} = \delta_{k\ell}, \quad \{\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}\} = 0, \quad \{\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\alpha)\dagger}\} = 0 \quad (1.2)$$

である。異なる成分間では、可換 :

$$[\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)\dagger}] = 0, \quad [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)}] = 0, \quad [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\beta)\dagger}] = 0 \quad (\alpha \neq \beta) \quad (1.3)$$

とする。パラボーズ粒子に拡張すると

$$[\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}] = \delta_{k\ell}, \quad [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}] = 0, \quad [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\alpha)\dagger}] = 0 \quad (1.4)$$

$$\{\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)\dagger}\} = 0, \quad \{\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)}\} = 0, \quad \{\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\beta)\dagger}\} = 0 \quad (\alpha \neq \beta) \quad (1.5)$$

である。両方同時に記述するには $[A, B]_\eta = AB + \eta BA$ ($\eta = \kappa$ or η for $\kappa = -\sigma = \pm 1$) を用いて

$$\left[\begin{array}{l} [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa = \delta_{k\ell}, \quad [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}]_\kappa = 0, \quad [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa = 0 \\ [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)\dagger}]_\sigma = 0, \quad [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\beta)}]_\sigma = 0, \quad [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\beta)\dagger}]_\sigma = 0 \quad (\alpha \neq \beta) \end{array} \right. \quad (\kappa = -\sigma = \pm 1) \quad (1.6)$$

と記述する。その時、 $[\hat{a}_\ell, \hat{a}_m^\dagger]_\kappa$ は

$$\begin{aligned} [\hat{a}_\ell, \hat{a}_m^\dagger]_\kappa &= \sum_{\alpha, \beta=1}^p [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\kappa = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)\dagger}]_\kappa + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\kappa \\ &= \sum_{\alpha=1}^p \delta_{\ell m} + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\kappa = p\delta_{\ell m} + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\kappa \end{aligned} \quad (1.7)$$

であるが、

$$\begin{aligned} \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\kappa &= \sum_{\alpha \neq \beta} (\hat{a}_\ell^{(\alpha)} \hat{a}_m^{(\beta)\dagger} + \kappa \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)}) = \sum_{\alpha \neq \beta} (\hat{a}_\ell^{(\alpha)} \hat{a}_m^{(\beta)\dagger} + \sigma \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)} - \sigma \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)} + \kappa \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)}) \\ &= \sum_{\alpha \neq \beta} \left(\cancel{[\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)\dagger}]_\sigma} + 2\kappa \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)} \right) = 2\kappa \sum_{\alpha \neq \beta} \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)} \end{aligned} \quad (1.8)$$

を用いて

$$[\hat{a}_\ell, \hat{a}_m^\dagger]_\kappa = p\delta_{lm} + 2\kappa \sum_{\alpha \neq \beta} \hat{a}_m^{(\beta)\dagger} \hat{a}_\ell^{(\alpha)} \quad (1.9)$$

を得る。もちろん、 $p=1$ であれば、 $\alpha = \beta = 1$ なので、正しく $[\hat{a}_\ell, \hat{a}_m^\dagger]_\kappa = \delta_{lm}$ を与える。

パラボーズ・パラフェルミ粒子の生成・消滅演算子の基本交換関係は

$$\begin{cases} [\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m]_\sigma] = 2\delta_{kl} \hat{a}_m \\ [\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m^\dagger]_\sigma] = 2\delta_{kl} \hat{a}_m^\dagger - 2\kappa\delta_{km} \hat{a}_\ell^\dagger \quad (\kappa = -\sigma = \pm 1) \\ [\hat{a}_k, [\hat{a}_\ell, \hat{a}_m]_\sigma] = 0 \end{cases} \quad (1.10)$$

であるので、(1.6)の関係の元で(1.1)は(1.10)を満たす必要がある。用いる公式は

$$[A, BC] = ABC + \eta BAC - \eta(BAC + \eta BCA) = [A, B]_C - \eta B[A, C]_\eta \quad (\eta = \kappa \text{ or } \sigma) \quad (1.11)$$

である。

$$[\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m]_\sigma] = 2\delta_{kl} \hat{a}_m$$

$$[\hat{a}_\ell^\dagger, \hat{a}_m]_\sigma = \sum_{\alpha, \beta=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\beta)}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\beta)}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma \quad (1.12)$$

を用いて

$$\begin{aligned} [\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m]_\sigma] &= \sum_{\beta=1}^p \sum_{\alpha=1}^p [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma] = \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma] + \sum_{\alpha \neq \beta} [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma] \\ &\Leftarrow [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma] = [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger} \hat{a}_m^{(\alpha)} + \sigma \hat{a}_m^{(\alpha)} \hat{a}_\ell^{(\alpha)\dagger}] \quad \Downarrow \sigma\kappa = -1 \\ &= [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa \hat{a}_m^{(\alpha)} - \kappa \hat{a}_\ell^{(\alpha)\dagger} \cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\kappa} + \sigma \left(\cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\kappa} \hat{a}_\ell^{(\alpha)\dagger} - \kappa \hat{a}_m^{(\alpha)} [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa \right) = 2\delta_{kl} \hat{a}_m^{(\alpha)} \\ &\Leftarrow [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)}]_\sigma] = [\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger} \hat{a}_m^{(\alpha)} + \sigma \hat{a}_m^{(\alpha)} \hat{a}_\ell^{(\alpha)\dagger}] \\ &= \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger}]_\sigma} \hat{a}_m^{(\alpha)} - \sigma \hat{a}_\ell^{(\alpha)\dagger} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)}]_\sigma} + \sigma \left(\cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)}]_\sigma} \hat{a}_\ell^{(\alpha)\dagger} - \sigma \hat{a}_m^{(\alpha)} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger}]_\sigma} \right) = 0 \\ &= \sum_{\alpha=1}^p 2\delta_{kl} \hat{a}_m^{(\alpha)} = 2\delta_{kl} \hat{a}_m \Rightarrow \text{OK!!!} \end{aligned} \quad (1.13)$$

$$[\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m^\dagger]_\sigma] = 2\delta_{kl} \hat{a}_m^\dagger - 2\kappa\delta_{km} \hat{a}_\ell^\dagger$$

$$[\hat{a}_\ell^\dagger, \hat{a}_m^\dagger]_\sigma = \sum_{\alpha, \beta=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\beta)\dagger}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\beta)\dagger}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma \quad (1.14)$$

を用いて

$$\begin{aligned}
[\hat{a}_k, [\hat{a}_\ell^\dagger, \hat{a}_m^\dagger]_\sigma] &= \sum_{\beta=1}^p \sum_{\alpha=1}^p [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma] = \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma] + \sum_{\alpha \neq \beta} [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma] \\
&\Leftrightarrow [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma] = [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger} \hat{a}_m^{(\alpha)\dagger} + \sigma \hat{a}_m^{(\alpha)\dagger} \hat{a}_\ell^{(\alpha)\dagger}] \\
&= [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa \hat{a}_m^{(\alpha)\dagger} - \kappa \hat{a}_\ell^{(\alpha)\dagger} [\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)\dagger}]_\kappa + \sigma \left([\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)\dagger}]_\kappa \hat{a}_\ell^{(\alpha)\dagger} - \kappa \hat{a}_m^{(\alpha)\dagger} [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)\dagger}]_\kappa \right) \\
&= 2\delta_{k\ell} \hat{a}_m^{(\alpha)} - 2\kappa \delta_{km} \hat{a}_\ell^{(\alpha)\dagger} \\
&\Leftrightarrow [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)\dagger}, \hat{a}_m^{(\alpha)\dagger}]_\sigma] = [\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger} \hat{a}_m^{(\alpha)\dagger} + \sigma \hat{a}_m^{(\alpha)\dagger} \hat{a}_\ell^{(\alpha)\dagger}] \\
&= \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger}]_\sigma} \hat{a}_m^{(\alpha)\dagger} - \sigma \hat{a}_\ell^{(\alpha)\dagger} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)\dagger}]_\sigma} + \sigma \left(\cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)\dagger}]_\sigma} \hat{a}_\ell^{(\alpha)\dagger} - \kappa \hat{a}_m^{(\alpha)\dagger} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)\dagger}]_\sigma} \right) = 0 \\
&= \sum_{\alpha=1}^p (2\delta_{k\ell} \hat{a}_m^{(\alpha)} - 2\kappa \delta_{km} \hat{a}_\ell^{(\alpha)\dagger}) = 2\delta_{k\ell} \hat{a}_m - 2\kappa \delta_{km} \hat{a}_\ell \Rightarrow \text{OK!!!} \tag{1.15}
\end{aligned}$$

$$[\hat{a}_k, [\hat{a}_\ell, \hat{a}_m]_\sigma] = 0$$

$$[\hat{a}_\ell, \hat{a}_m]_\sigma = \sum_{\alpha, \beta=1}^p [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma + \sum_{\alpha \neq \beta} [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\beta)}]_\sigma = \sum_{\alpha=1}^p [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma \tag{1.16}$$

を用いて

$$\begin{aligned}
[\hat{a}_k, [\hat{a}_\ell, \hat{a}_m]_\sigma] &= \sum_{\beta=1}^p \sum_{\alpha=1}^p [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma] = \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma] + \sum_{\alpha \neq \beta} [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma] \\
&\Leftrightarrow [\hat{a}_k^{(\alpha)}, [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma] = [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)} \hat{a}_m^{(\alpha)} + \sigma \hat{a}_m^{(\alpha)} \hat{a}_\ell^{(\alpha)}] \\
&= \cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}]_\kappa} \hat{a}_m^{(\alpha)} - \kappa \hat{a}_\ell^{(\alpha)} \cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\kappa} + \sigma \left(\cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\kappa} \hat{a}_\ell^{(\alpha)} - \kappa \hat{a}_m^{(\alpha)} \cancel{[\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}]_\kappa} \right) = 0 \\
&\Leftrightarrow [\hat{a}_k^{(\beta)}, [\hat{a}_\ell^{(\alpha)}, \hat{a}_m^{(\alpha)}]_\sigma] = [\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)} \hat{a}_m^{(\alpha)} + \sigma \hat{a}_m^{(\alpha)} \hat{a}_\ell^{(\alpha)}] \\
&= \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)}]_\sigma} \hat{a}_m^{(\alpha)} - \sigma \hat{a}_\ell^{(\alpha)} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)}]_\sigma} + \sigma \left(\cancel{[\hat{a}_k^{(\beta)}, \hat{a}_m^{(\alpha)}]_\sigma} \hat{a}_\ell^{(\alpha)} - \sigma \hat{a}_m^{(\alpha)} \cancel{[\hat{a}_k^{(\beta)}, \hat{a}_\ell^{(\alpha)}]_\sigma} \right) = 0 \\
&= 0 \Rightarrow \text{OK!!!} \tag{1.17}
\end{aligned}$$

グリーン成分による記述

生成・消滅演算子から構成される生成子はグリーン成分では、(1.12)、(1.14)と(1.16)から

$$\begin{cases} L_{kl} = \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm} = \frac{1}{2} \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\alpha)\dagger}]_\sigma \left(\equiv \sum_{\alpha=1}^p L_{kl}^{(\alpha)} \right) \\ M_{kl} = \frac{1}{2} [\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm} = \frac{1}{2} \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)}, \hat{a}_\ell^{(\alpha)}]_\sigma \left(\equiv \sum_{\alpha=1}^p M_{kl}^{(\alpha)} \right) \\ N_{kl} = \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} = \frac{1}{2} \sum_{\alpha=1}^p [\hat{a}_k^{(\alpha)\dagger}, \hat{a}_\ell^{(\alpha)}]_\sigma \left(\equiv \sum_{\alpha=1}^p N_{kl}^{(\alpha)} \right) \end{cases} \tag{1.18}$$

を得る。