

# 付録 D ボゴリューボフ変換とフレーバー

## $f$ 個のフレーバーの生成・消滅演算子

生成・消滅演算子の  $\hat{a}$  と  $\hat{a}^\dagger$  のユニタリー変換  $U$  は

- **パラボーズ粒子** ( $[\hat{a}, \hat{a}^\dagger] = 1$  に依存しないボーズ粒子) に対して

$$\begin{aligned} \hat{a}' &= U^\dagger \hat{a} U \simeq (1 - i\alpha) \hat{a} - (\theta + i\varepsilon) \hat{a}^\dagger \Leftrightarrow [\hat{a}', \hat{a}'^\dagger] = [\hat{a}, \hat{a}^\dagger] \\ U &\simeq I - i \left( \alpha N + \frac{1}{2} (\varepsilon + i\theta) M + \frac{1}{2} (\varepsilon - i\theta) L \right) \quad (N^\dagger = N, L^\dagger = M) \\ [N, \hat{a}] &= -\hat{a}, \quad [L, \hat{a}] = -2\hat{a}^\dagger, \quad [M, \hat{a}] = 0 \end{aligned} \quad (1.1)$$

$$\begin{aligned} N &= \frac{1}{2} \{\hat{a}^\dagger, \hat{a}\}, \quad L = \frac{1}{2} \{\hat{a}^\dagger, \hat{a}^\dagger\}, \quad M = \frac{1}{2} \{\hat{a}, \hat{a}\} \\ [\hat{a}, \{\hat{a}^\dagger, \hat{a}\}] &= 2\hat{a}, \quad [\hat{a}, \{\hat{a}^\dagger, \hat{a}^\dagger\}] = 4\hat{a}^\dagger, \quad [\hat{a}, \{\hat{a}, \hat{a}\}] = 0 \end{aligned} \quad (1.2)$$

- **パラフェルミ粒子** ( $\{\hat{a}, \hat{a}^\dagger\} = 1$  に依存しないフェルミ粒子) に対して

$$\begin{aligned} \hat{a}' &= U^\dagger \hat{a} U \simeq (1 - i\alpha) \hat{a} \Leftrightarrow \{\hat{a}', \hat{a}'^\dagger\} = \{\hat{a}, \hat{a}^\dagger\} \\ U &\simeq I - i\alpha N \quad (N^\dagger = N) \\ [N, \hat{a}] &= -\hat{a} \end{aligned} \quad (1.3)$$

$$N = \frac{1}{2} [\hat{a}^\dagger, \hat{a}], \quad [\hat{a}, [\hat{a}^\dagger, \hat{a}]] = 2\hat{a}, \quad [\hat{a}, [\hat{a}^\dagger, \hat{a}^\dagger]] = 0, \quad [\hat{a}, [\hat{a}, \hat{a}]] = 0 \quad (1.4)$$

である。 $f$  個の組の  $\hat{a}$  と  $\hat{a}^\dagger$  を  $\hat{a}_k, \hat{a}_k^\dagger$  ( $k=1 \dots f$ ) とするとき、(1.1)と(1.3)の

- 演算子  $\hat{a}, \hat{a}^\dagger$  をそれぞれ  $f$  行の列ベクトル

- 係数  $(\alpha, \theta, \varepsilon)$  をそれぞれ  $f$  行  $f$  列の正方行列(実数条件:  $\alpha_{lm}^* = \alpha_{lm}, \theta_{lm}^* = \theta_{lm}, \varepsilon_{lm}^* = \varepsilon_{lm}$ )

- 演算子  $(N, L, M)$  をそれぞれ  $f$  行  $f$  列の正方行列

と見做せばよい。従って、 $f$  個の組の  $\hat{a}$  と  $\hat{a}^\dagger$  を  $\hat{a}_k, \hat{a}_k^\dagger$  ( $k=1 \dots f$ ) とするとき、(1.1)と(1.3)は

- **パラボーズ粒子・パラフェルミ粒子** に対して

$$\begin{aligned} \hat{a}'_k &= U^\dagger \hat{a}_k U = \sum_{m=1}^f \left[ (I - i\alpha)_{km} \hat{a}_m - (\theta + i\varepsilon)_{km} \hat{a}_m^\dagger \right] \\ U &= I - i \sum_{\ell, m=1}^f \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2} (\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2} (\varepsilon - i\theta)_{m\ell} L_{\ell m} \right) \end{aligned} \quad (1.5)$$

(\*)ここで、(1.3)を(1.1)に拡張しておく。(1.3)に於て対応する  $M$  は  $M = [\hat{a}, \hat{a}]/2 = 0$  (や  $L$ )なので、(1.3)に

$M$  (や  $L$ )は無いが、フレーバーがあると  $M_{k\ell} = [\hat{a}_k, \hat{a}_\ell]/2 \neq 0$  (や  $L_{k\ell} = [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]/2 \neq 0$ ) が可能になる。

とする。(1.5)より、 $\hat{a}_k^\dagger$  と  $U^\dagger$  は  $\alpha_{\ell m}^* = \alpha_{\ell m}$  等に注意して

$$\begin{aligned}
\hat{a}_k^\dagger &= \left( \sum_{m=1}^f [(I - i\alpha)_{km} \hat{a}_m - (\theta + i\varepsilon)_{km} \hat{a}_m^\dagger] \right)^\dagger = \sum_{m=1}^f [(I + i\alpha^*)_{km} \hat{a}_m^\dagger - (\theta^* - i\varepsilon^*)_{km} \hat{a}_m] \\
&= \sum_{m=1}^f [(I + i\alpha)_{km} \hat{a}_m^\dagger - (\theta - i\varepsilon)_{km} \hat{a}_m] \\
U^\dagger &= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m} + \frac{1}{2} (\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2} (\varepsilon - i\theta)_{m\ell} L_{\ell m} \right)^\dagger \\
&= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m}^* N_{\ell m}^\dagger + \frac{1}{2} (\varepsilon^* - i\theta^*)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2} (\varepsilon^* + i\theta^*)_{\ell m} L_{\ell m}^\dagger \right) \\
&= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2} (\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2} (\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \right)
\end{aligned} \tag{1.6}$$

を得る。従って

$$\boxed{
\begin{aligned}
\hat{a}_k^\dagger &= \sum_{m=1}^f [(I + i\alpha)_{mk} \hat{a}_m^\dagger - (\theta - i\varepsilon)_{mk} \hat{a}_m] \\
U^\dagger &= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2} (\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2} (\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \right)
\end{aligned}
} \tag{1.7}$$

まず、ボーズ粒子・フェルミ粒子の交換関係： $[\hat{a}'_k, \hat{a}'_\ell] = \delta_{k\ell}$ 、 $[\hat{a}'_k, \hat{a}'_\ell] = 0$  や  $\{\hat{a}'_k, \hat{a}'_\ell\} = \delta_{k\ell}$ 、

$\{\hat{a}'_k, \hat{a}'_\ell\} = 0$  からの条件を調べるために、 $[A, B]_- = AB - BA = [A, B]$ ,  $[A, B]_+ = AB + BA = \{A, B\}$  の統一記法

$$\boxed{[A, B]_\kappa = AB + \kappa BA \quad (\kappa = \pm 1)} \tag{1.8}$$

とする。その時、 $[\hat{a}'_k, \hat{a}'_\ell]_{\kappa=\pm} = \delta_{k\ell}$ ,  $[\hat{a}'_k, \hat{a}'_\ell]_{\kappa=\pm} = 0$  から

$$\begin{aligned}
\odot \delta_{k\ell} &= [\hat{a}'_k, \hat{a}'_\ell]_\kappa = \left[ \sum_{m=1}^f [(I - i\alpha)_{km} \hat{a}_m - (\theta + i\varepsilon)_{km} \hat{a}_m^\dagger], \sum_{n=1}^f [(I + i\alpha^*)_{n\ell} \hat{a}_n^\dagger - (\theta^* - i\varepsilon^*)_{n\ell} \hat{a}_n] \right]_\kappa \\
&= \left[ \sum_{m=1}^f (I - i\alpha)_{km} \hat{a}_m, \sum_{n=1}^f (I + i\alpha^*)_{n\ell} \hat{a}_n^\dagger \right]_\kappa = \sum_{m, n=1}^f (I - i\alpha)_{km} (I + i\alpha^*)_{n\ell} [\hat{a}_m, \hat{a}_n^\dagger]_\kappa \\
&= \sum_{m, n=1}^f (\delta_{km} \delta_{n\ell} - i\alpha_{km} \delta_{n\ell} + i\delta_{km} \alpha_{n\ell}^*) \delta_{mn} = \delta_{k\ell} - i(\alpha_{k\ell} - \alpha_{k\ell}^*) \stackrel{\alpha_{k\ell}^* = \alpha_{k\ell}}{=} \delta_{k\ell} \Rightarrow \text{OK!!!}
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
\odot 0 &= [\hat{a}'_k, \hat{a}'_\ell]_\kappa = \left[ \sum_{m=1}^f [(I - i\alpha)_{km} \hat{a}_m - (\theta + i\varepsilon)_{km} \hat{a}_m^\dagger], \sum_{n=1}^f [(I - i\alpha)_{\ell n} \hat{a}_n - (\theta + i\varepsilon)_{\ell n} \hat{a}_n^\dagger] \right]_\kappa \\
&= \left[ \sum_{m=1}^f (I - i\alpha)_{km} \hat{a}_m, \sum_{n=1}^f -(\theta + i\varepsilon)_{\ell n} \hat{a}_n^\dagger \right]_\kappa + \left[ \sum_{m=1}^f [-(\theta + i\varepsilon)_{km} \hat{a}_m^\dagger], \sum_{n=1}^f [(I - i\alpha)_{\ell n} \hat{a}_n] \right]_\kappa \\
&\simeq - \sum_{m, n=1}^f (\delta_{km} (\theta + i\varepsilon)_{\ell n} [\hat{a}_m, \hat{a}_n^\dagger]_\kappa + (\theta + i\varepsilon)_{km} \delta_{\ell n} [\hat{a}_m^\dagger, \hat{a}_n]_\kappa)
\end{aligned} \tag{1.10}$$

$$= - \sum_{m,n=1}^f \left( \delta_{km} (\theta + i\varepsilon)_{\ell m} + \kappa (\theta + i\varepsilon)_{km} \delta_{\ell m} \right) \simeq - \left( (\theta + i\varepsilon)_{\ell k} + \kappa (\theta + i\varepsilon)_{k\ell} \right) = 0$$

より  $(\theta + i\varepsilon)_{\ell k} + \kappa (\theta + i\varepsilon)_{k\ell} = 0$  がわかる。その結果、(1.5)の  $U$  より

$$(\varepsilon + i\theta)_{\ell m} M_{\ell m} = \frac{(\varepsilon + i\theta)_{\ell m} - \kappa (\theta + i\varepsilon)_{m\ell}}{2} M_{\ell m} \Leftarrow (\theta + i\varepsilon)_{\ell k} + \kappa (\theta + i\varepsilon)_{k\ell} = 0 \quad (\kappa = \pm 1) \quad (1.11)$$

を用いると、 $M_{\ell m} = \kappa M_{m\ell}$  の時に

$$(\varepsilon + i\theta)_{\ell m} M_{\ell m} = \frac{(\varepsilon + i\theta)_{\ell m} - \kappa (\theta + i\varepsilon)_{m\ell}}{2} M_{\ell m} = \frac{(\varepsilon + i\theta)_{\ell m} M_{\ell m} - \kappa (\theta + i\varepsilon)_{m\ell} \kappa M_{m\ell}}{2} = 0 \quad (1.12)$$

となる。 $U$  に現れる  $M_{\ell m}$  には  $M_{\ell m} = -\kappa M_{m\ell}$  ( $\Rightarrow M_{\ell m} \pm M_{m\ell} = 0$ ) の制限が掛かる。以上から

$$\boxed{\begin{array}{l} (\theta + i\varepsilon)_{\ell k} + \kappa (\theta + i\varepsilon)_{k\ell} = 0 \quad (\Rightarrow (\theta + i\varepsilon)_{\ell k} = -\kappa (\theta + i\varepsilon)_{k\ell}) \\ M_{\ell m} + \kappa M_{m\ell} = 0 \quad (\Rightarrow M_{\ell m} = -\kappa M_{m\ell}) \text{ for } [\hat{a}_k, \hat{a}_\ell^\dagger]_{\kappa=\pm 1} = \delta_{k\ell} \end{array}} \quad (1.13)$$

である。以上が、**ボーズ粒子**と**フェルミ粒子**の場合である。

## ユニタリー演算子

$$[\hat{a}_k, \hat{a}_\ell^\dagger]_{\kappa} = \delta_{k\ell}, \quad [\hat{a}_k, \hat{a}_\ell]_{\kappa} = 0, \quad [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\kappa} = 0 \text{ に依らない結果}$$

**パラボーズ粒子**と**パラフェルミ粒子**に拡張するため、以降、 $[\hat{a}_k, \hat{a}_\ell^\dagger]_{\kappa} = \delta_{k\ell}$ 、 $[\hat{a}_k, \hat{a}_\ell]_{\kappa} = 0$ 、

$[\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\kappa} = 0$  の条件は使用せずに、(1.5)、(1.7)と(1.13)のみ使用する。従って、条件は

$$\bullet \quad U^\dagger \hat{a}_k U = \hat{a}'_k, \quad U^\dagger \hat{a}_k^\dagger U = \hat{a}'_k^\dagger$$

になる。これらから、ユニタリー演算子  $N_{\ell m}, M_{\ell m}, L_{\ell m}$  の性質が求められる。

### $U^\dagger U = I$ を要請

(1.7)より

$$\begin{aligned} I = U^\dagger U &= \left[ I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2} (\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2} (\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \right) \right] \\ &\quad \cdot \left[ I - i \sum_{\ell, m=1}^f \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2} (\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2} (\varepsilon - i\theta)_{m\ell} L_{\ell m} \right) \right] \\ &\simeq I + i \sum_{\ell, m=1}^f \left\{ \begin{array}{l} \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2} (\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2} (\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \\ - \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2} (\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2} (\varepsilon - i\theta)_{m\ell} L_{\ell m} \right) \end{array} \right\} \end{aligned} \quad (1.14)$$

を得る。(1.14)に於て

$$\begin{aligned}
& \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \\
& - \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2}(\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2}(\varepsilon - i\theta)_{m\ell} L_{\ell m} \right) = 0 \\
& \Rightarrow N_{m\ell}^\dagger = N_{\ell m}, M_{m\ell}^\dagger = L_{\ell m}, L_{m\ell}^\dagger = M_{\ell m}
\end{aligned} \tag{1.15}$$

である。これより

$$\begin{aligned}
U^\dagger &= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m}^\dagger + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} M_{\ell m}^\dagger + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} L_{\ell m}^\dagger \right) \\
&= I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{m\ell} + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} L_{m\ell} + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} M_{m\ell} \right)
\end{aligned} \tag{1.16}$$

より

$$\boxed{U^\dagger = I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{m\ell} + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} L_{m\ell} + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} M_{m\ell} \right)} \tag{1.17}$$

である。

$\hat{a}'_k = U^\dagger \hat{a}_k U$  を要請

(1.5)より、 $L, M$  及び  $N$  への条件が現れる。つまり、

$$\begin{aligned}
U^\dagger \hat{a}_k U &= \sum_{m=1}^f \left[ (I - i\alpha)_{km} \hat{a}_m - (\theta + i\varepsilon)_{km} \hat{a}_m^\dagger \right] \\
&= \left[ I + i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{m\ell} + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} L_{m\ell} + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} M_{m\ell} \right) \right] \hat{a}_k \\
&\quad \cdot \left[ I - i \sum_{\ell, m=1}^f \left( \alpha_{\ell m} N_{\ell m} + \frac{1}{2}(\varepsilon + i\theta)_{\ell m} M_{\ell m} + \frac{1}{2}(\varepsilon - i\theta)_{\ell m} L_{\ell m} \right) \right] \\
&= \hat{a}_k + i \sum_{\ell, m=1}^f \left\{ \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2}(\varepsilon - i\theta)_{m\ell} L_{\ell m} + \frac{1}{2}(\varepsilon + i\theta)_{m\ell} M_{\ell m} \right) \hat{a}_k \right. \\
&\quad \left. - \hat{a}_k \left( \alpha_{m\ell} N_{\ell m} + \frac{1}{2}(\varepsilon + i\theta)_{m\ell} M_{\ell m} + \frac{1}{2}(\varepsilon - i\theta)_{m\ell} L_{\ell m} \right) \right\} \\
&= \hat{a}_k + i \sum_{\ell, m=1}^f \left( [N_{\ell m}, \hat{a}_k] \alpha_{m\ell} + \frac{1}{2} [M_{\ell m}, \hat{a}_k] i(\theta - i\varepsilon)_{m\ell} - \frac{1}{2} [L_{\ell m}, \hat{a}_k] i(\theta + i\varepsilon)_{m\ell} \right)
\end{aligned} \tag{1.18}$$

である。両辺を比較して

$$\begin{aligned}
\odot -i\alpha_{km} \hat{a}_m &= i [N_{\ell m}, \hat{a}_k] \alpha_{\ell m} \Rightarrow [N_{\ell m}, \hat{a}_k] = -\delta_{\ell k} \hat{a}_m \\
\odot -(\theta + i\varepsilon)_{km} \hat{a}_m^\dagger &= i \left( -\frac{1}{2} \right) [L_{\ell m}, \hat{a}_k] i(\theta + i\varepsilon)_{\ell m} \Rightarrow -(\theta + i\varepsilon)_{km} \hat{a}_m^\dagger = \frac{1}{2} [L_{\ell m}, \hat{a}_k] (\theta + i\varepsilon)_{\ell m} \\
&\Leftrightarrow (\theta + i\varepsilon)_{\ell k} + \kappa(\theta + i\varepsilon)_{k\ell} = 0 \\
&\Rightarrow [-(\theta + i\varepsilon)_{km} + \kappa(\theta + i\varepsilon)_{mk}] \hat{a}_m^\dagger = -(\theta + i\varepsilon)_{km} \hat{a}_m^\dagger + \kappa(\theta + i\varepsilon)_{\ell k} \hat{a}_\ell^\dagger = [L_{\ell m}, \hat{a}_k] (\theta + i\varepsilon)_{\ell m} \\
&\Rightarrow [L_{\ell m}, \hat{a}_k] = -\delta_{\ell k} \hat{a}_m^\dagger + \kappa \delta_{mk} \hat{a}_\ell^\dagger \rightarrow [\hat{a}_k^\dagger, L_{\ell m}^\dagger] = -\delta_{\ell k} \hat{a}_m + \kappa \delta_{mk} \hat{a}_\ell \\
&\quad \rightarrow [M_{m\ell}, \hat{a}_k^\dagger] = \delta_{\ell k} \hat{a}_m - \kappa \delta_{mk} \hat{a}_\ell \rightarrow [M_{\ell m}, \hat{a}_k^\dagger] = \delta_{mk} \hat{a}_\ell - \kappa \delta_{\ell k} \hat{a}_m
\end{aligned}$$

$$\odot 0 = \frac{1}{2} [M_{\ell m}, \hat{a}_k] \Rightarrow [M_{\ell m}, \hat{a}_k] = 0 \quad (1.19)$$

を得る。以上から、(1.13)と(1.15)と共にまとめると

$$\boxed{N_{m\ell}^\dagger = N_{\ell m}, M_{m\ell}^\dagger = L_{\ell m}, L_{m\ell}^\dagger = M_{\ell m}} \quad (1.20)$$

$$\boxed{L_{\ell m} + \kappa L_{m\ell} = 0, M_{\ell m} + \kappa M_{m\ell} = 0}$$

$$\boxed{\begin{aligned} [N_{\ell m}, \hat{a}_k] &= -\delta_{\ell k} \hat{a}_m, [N_{\ell m}, \hat{a}_k^\dagger] = \delta_{mk} \hat{a}_\ell^\dagger \\ [L_{\ell m}, \hat{a}_k] &= -(\delta_{\ell k} \hat{a}_m^\dagger - \kappa \delta_{mk} \hat{a}_\ell^\dagger) \\ [M_{\ell m}, \hat{a}_k^\dagger] &= \delta_{mk} \hat{a}_\ell - \kappa \delta_{\ell k} \hat{a}_m \\ [L_{\ell m}, \hat{a}_k^\dagger] &= [M_{\ell m}, \hat{a}_k] = 0 \end{aligned}} \quad (1.21)$$

を得る。

## Lie 代数の交換関係

(1.20)に、ヤコビ恒等式(Jacobi identity):

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \quad (1.22)$$

を用いると、3つの演算子間の交換関係として

$$\begin{aligned} \odot [N_{k\ell}, [N_{mn}, \hat{a}_j]] + [N_{mn}, [\hat{a}_j, N_{k\ell}]] + [\hat{a}_j, [N_{k\ell}, N_{mn}]] &= 0 \\ \Leftrightarrow [N_{mn}, \hat{a}_j] &= -\delta_{mj} \hat{a}_n, [N_{k\ell}, \hat{a}_j] = -\delta_{kj} \hat{a}_\ell \\ [[N_{k\ell}, N_{mn}], \hat{a}_j] &= [N_{k\ell}, -\delta_{mj} \hat{a}_n] + [N_{mn}, -(-\delta_{kj} \hat{a}_\ell)] = -\delta_{mj} (-\delta_{kn} \hat{a}_\ell) + \delta_{kj} (-\delta_{m\ell} \hat{a}_n) \\ &= -\delta_{kn} [N_{m\ell}, \hat{a}_j] + \delta_{m\ell} [N_{kn}, \hat{a}_j] \stackrel{m \leftrightarrow n}{\Rightarrow} [\delta_{m\ell} N_{kn} - \delta_{kn} N_{m\ell}, \hat{a}_j] \Rightarrow [N_{k\ell}, N_{mn}] = \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell} \end{aligned} \quad (1.23)$$

$$\begin{aligned} \odot [N_{mn}, [L_{k\ell}, \hat{a}_j]] + [L_{k\ell}, [\hat{a}_j, N_{mn}]] + [\hat{a}_j, [N_{mn}, L_{k\ell}]] &= 0 \\ \Leftrightarrow [L_{k\ell}, \hat{a}_j] &= -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger), [N_{mn}, \hat{a}_j] = -\delta_{mj} \hat{a}_n \\ [[N_{mn}, L_{k\ell}], \hat{a}_j] &= [N_{mn}, -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger)] + [L_{k\ell}, -(-\delta_{mj} \hat{a}_n)] \\ &= -\delta_{kj} [N_{mn}, \hat{a}_\ell^\dagger] + \kappa \delta_{\ell j} [N_{mn}, \hat{a}_k^\dagger] + \delta_{mj} [L_{k\ell}, \hat{a}_n] \\ \Leftrightarrow [N_{mn}, \hat{a}_\ell^\dagger] &= \delta_{n\ell} \hat{a}_m^\dagger, [L_{k\ell}, \hat{a}_n] = -(\delta_{kn} \hat{a}_\ell^\dagger - \kappa \delta_{\ell n} \hat{a}_k^\dagger) \\ &= -\delta_{kj} (\delta_{n\ell} \hat{a}_m^\dagger) + \kappa \delta_{\ell j} (\delta_{nk} \hat{a}_m^\dagger) - \delta_{mj} (\delta_{kn} \hat{a}_\ell^\dagger - \kappa \delta_{\ell n} \hat{a}_k^\dagger) \\ &= \delta_{\ell n} (-\delta_{kj} \hat{a}_m^\dagger + \kappa \delta_{mj} \hat{a}_k^\dagger) + \delta_{nk} (\kappa \delta_{\ell j} \hat{a}_m^\dagger - \delta_{mj} \hat{a}_\ell^\dagger) \\ \Leftrightarrow [L_{mk}, \hat{a}_j] &= -(\delta_{mj} \hat{a}_k^\dagger - \kappa \delta_{kj} \hat{a}_m^\dagger), [L_{m\ell}, \hat{a}_j] = -(\delta_{mj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_m^\dagger) \\ &= \delta_{\ell n} (-\kappa) [L_{mk}, \hat{a}_j] + \delta_{nk} [L_{m\ell}, \hat{a}_j] = [-\kappa \delta_{\ell n} L_{mk} + \delta_{nk} L_{m\ell}, \hat{a}_j] \Leftrightarrow L_{\ell m} + \kappa L_{m\ell} = 0 \\ [N_{mn}, L_{k\ell}] &= -\kappa \delta_{\ell n} L_{mk} + \delta_{nk} L_{m\ell} \Rightarrow [L_{k\ell}, N_{mn}] = -\delta_{nk} L_{m\ell} + \kappa \delta_{\ell n} L_{mk} \end{aligned} \quad (1.24)$$

$$\begin{aligned} \odot [L_{k\ell}, N_{mn}] &= -\delta_{nk} L_{m\ell} + \kappa \delta_{\ell n} L_{mk} \Rightarrow [N_{mn}^\dagger, L_{k\ell}^\dagger] = -\delta_{nk} L_{m\ell}^\dagger + \kappa \delta_{\ell n} L_{mk}^\dagger \Leftarrow L_{m\ell}^\dagger = M_{\ell m} \\ [N_{nm}, M_{\ell k}] &= -\delta_{nk} M_{\ell m} + \kappa \delta_{\ell n} M_{km} \Rightarrow [M_{k\ell}, N_{mn}] = \delta_{mk} M_{\ell n} - \kappa \delta_{\ell m} M_{kn} \Leftarrow M_{\ell m} = -\kappa M_{m\ell} \end{aligned} \quad (1.25)$$

$$\begin{aligned} \odot [M_{mn}, [L_{k\ell}, \hat{a}_j]] + [L_{k\ell}, \cancel{[\hat{a}_j, M_{mn}]}] + [\hat{a}_j, [M_{mn}, L_{k\ell}]] &= 0 \Leftarrow [L_{k\ell}, \hat{a}_j] = -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger) \\ [[M_{mn}, L_{k\ell}], \hat{a}_j] &= [M_{mn}, -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger)] = -\delta_{kj} [M_{mn}, \hat{a}_\ell^\dagger] + \kappa \delta_{\ell j} [M_{mn}, \hat{a}_k^\dagger] \\ &\Leftarrow [M_{mn}, \hat{a}_\ell^\dagger] = \delta_{n\ell} \hat{a}_m - \kappa \delta_{m\ell} \hat{a}_n \\ &= -\delta_{kj} (\delta_{n\ell} \hat{a}_m - \kappa \delta_{m\ell} \hat{a}_n) + \kappa \delta_{\ell j} (\delta_{nk} \hat{a}_m - \kappa \delta_{mk} \hat{a}_n) \\ &= -\delta_{n\ell} \delta_{kj} \hat{a}_m + \kappa \delta_{m\ell} \delta_{kj} \hat{a}_n + \kappa \delta_{nk} \delta_{\ell j} \hat{a}_m - \delta_{mk} \delta_{\ell j} \hat{a}_n \Leftarrow [N_{\ell m}, \hat{a}_j] = -\delta_{\ell j} \hat{a}_m \\ &= \delta_{\ell n} [N_{km}, \hat{a}_j] - \kappa \delta_{\ell m} [N_{kn}, \hat{a}_j] - \kappa \delta_{kn} [N_{\ell m}, \hat{a}_j] + \delta_{km} [N_{\ell n}, \hat{a}_j] \\ &= [\delta_{\ell n} N_{km} - \kappa \delta_{\ell m} N_{kn} - \kappa \delta_{kn} N_{\ell m} + \delta_{km} N_{\ell n}, \hat{a}_j] \\ \Rightarrow [M_{mn}, L_{k\ell}] &= \delta_{\ell n} N_{km} - \kappa \delta_{\ell m} N_{kn} - \kappa \delta_{kn} N_{\ell m} + \delta_{km} N_{\ell n} \\ \Rightarrow [L_{k\ell}, M_{mn}] &= -\delta_{km} N_{\ell n} + \kappa \delta_{kn} N_{\ell m} - \delta_{\ell n} N_{km} + \kappa \delta_{\ell m} N_{kn} \end{aligned} \quad (1.26)$$

$$\begin{aligned} \odot [L_{mn}, [L_{k\ell}, \hat{a}_j]] + [L_{k\ell}, [\hat{a}_j, L_{mn}]] + [\hat{a}_j, [L_{mn}, L_{k\ell}]] &= 0 \Leftarrow [L_{k\ell}, \hat{a}_j] = -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger) \\ [[L_{mn}, L_{k\ell}], \hat{a}_j] &= [L_{mn}, -(\delta_{kj} \hat{a}_\ell^\dagger - \kappa \delta_{\ell j} \hat{a}_k^\dagger)] + [L_{k\ell}, \delta_{nj} \hat{a}_n^\dagger - \kappa \delta_{nj} \hat{a}_m^\dagger] = 0 \Leftarrow [L_{k\ell}, \hat{a}_j^\dagger] = 0 \\ [L_{mn}, L_{k\ell}] &= 0 \Rightarrow [M_{mn}, M_{k\ell}] = 0 \Leftarrow M_{mn}^\dagger = L_{nm} \end{aligned} \quad (1.27)$$

より

$$\begin{aligned} [N_{k\ell}, N_{mn}] &= \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell}, \quad N_{m\ell}^\dagger = N_{\ell m} \\ [L_{k\ell}, N_{mn}] &= -\delta_{nk} L_{m\ell} + \kappa \delta_{\ell n} L_{mk}, \quad [M_{k\ell}, N_{mn}] = \delta_{mk} M_{\ell n} - \kappa \delta_{\ell m} M_{kn} \\ [L_{k\ell}, M_{mn}] &= -\delta_{km} N_{\ell n} + \kappa \delta_{kn} N_{\ell m} - \delta_{\ell n} N_{km} + \kappa \delta_{\ell m} N_{kn} \\ [L_{mn}, L_{k\ell}] &= [M_{mn}, M_{k\ell}] = 0, \quad M_{m\ell}^\dagger = L_{\ell m}, \quad L_{m\ell}^\dagger = M_{\ell m} \\ (*) \quad L_{mn} + \kappa L_{nm} &= 0, \quad M_{mn} + \kappa M_{nm} = 0 \end{aligned} \quad (1.28)$$

を得る。

## Lie 代数と生成・消滅演算子

フレーバーが 1 ( $\neq 1$ ) のボーズ粒子の場合に

$$L = \frac{1}{2} \{ \hat{a}^\dagger, \hat{a}^\dagger \}, \quad M = \frac{1}{2} \{ \hat{a}, \hat{a} \}, \quad N = \frac{1}{2} \{ \hat{a}^\dagger, \hat{a} \} \quad (1.29)$$

であったので、生成・消滅演算子では、 $\eta_{L,M,N}$  を定数として

$$L_{k\ell} = \frac{1}{2} \eta_L [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, \quad M_{k\ell} = \frac{1}{2} \eta_M [\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, \quad N_{k\ell} = \frac{1}{2} \eta_N [\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (1.30)$$

と類推できる。ここに

- $\sigma = 1 \cdots$  ボーズ粒子、 $\sigma = -1 \cdots$  フェルミ粒子  $\Rightarrow \boxed{\sigma = -\kappa}$

に注意する。この設定が  $L_{k\ell}, M_{k\ell}, N_{k\ell}$  の満たす関係式を再現する。実際、

$$\begin{aligned}
L_{k\ell}^\dagger &= \frac{1}{2}\eta_L \left( [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_\sigma \right)^\dagger = \frac{1}{2}\eta_L [\hat{a}_\ell, \hat{a}_k]_\sigma = M_{\ell k} \\
M_{k\ell}^\dagger &= \frac{1}{2}\eta_M \left( [\hat{a}_k, \hat{a}_\ell]_\sigma \right)^\dagger = \frac{1}{2}\eta_M [\hat{a}_\ell^\dagger, \hat{a}_k^\dagger]_\sigma = M_{\ell k} \\
N_{k\ell}^\dagger &= \frac{1}{2}\eta_N \frac{1}{2} \left( [\hat{a}_k^\dagger, \hat{a}_\ell]_\sigma \right)^\dagger = \frac{1}{2}\eta_N [\hat{a}_\ell^\dagger, \hat{a}_k]_\sigma = N_{\ell k}
\end{aligned} \tag{1.31}$$

より (1.20) の  $N_{m\ell}^\dagger = N_{\ell m}$ ,  $M_{m\ell}^\dagger = L_{\ell m}$ ,  $L_{m\ell}^\dagger = M_{\ell m}$  を満たし

$$\begin{aligned}
L_{k\ell} &= \frac{1}{2}\eta_L [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_\sigma = \sigma \frac{1}{2}\eta_L [\hat{a}_\ell^\dagger, \hat{a}_k^\dagger]_\sigma = -\kappa L_{\ell k} \\
M_{k\ell} &= \frac{1}{2}\eta_M [\hat{a}_k, \hat{a}_\ell]_\sigma = \sigma \frac{1}{2}\eta_M [\hat{a}_\ell, \hat{a}_k]_\sigma = -\kappa M_{\ell k}
\end{aligned} \tag{1.32}$$

より、(1.20) の  $L_{\ell m} + \kappa L_{m\ell} = 0$ ,  $M_{\ell m} + \kappa M_{m\ell} = 0$  を導く。最後に、(1.28) は、(1.21) を用いれば

$$\begin{aligned}
\odot [N_{k\ell}, N_{mn}] &= \left[ N_{k\ell}, \eta_N \frac{1}{2} [\hat{a}_m^\dagger, \hat{a}_n]_{\sigma=\pm} \right] = \frac{\eta_N}{2} [N_{k\ell}, \hat{a}_m^\dagger \hat{a}_n + \sigma \hat{a}_n \hat{a}_m^\dagger] \\
&= \frac{\eta_N}{2} \left( [N_{k\ell}, \hat{a}_m^\dagger] \hat{a}_n + \hat{a}_m^\dagger [N_{k\ell}, \hat{a}_n] + \sigma \left( [N_{k\ell}, \hat{a}_n] \hat{a}_m^\dagger + \hat{a}_n [N_{k\ell}, \hat{a}_m^\dagger] \right) \right) \\
&= \frac{\eta_N}{2} \left( \left[ [N_{k\ell}, \hat{a}_m^\dagger], \hat{a}_n \right]_\sigma + \sigma \left[ [N_{k\ell}, \hat{a}_n], \hat{a}_m^\dagger \right]_\sigma \right) = \frac{\eta_N}{2} \left( [\delta_{\ell m} \hat{a}_k^\dagger, \hat{a}_n]_\sigma + [\hat{a}_m^\dagger, -\delta_{kn} \hat{a}_\ell]_\sigma \right) \\
&= \delta_{\ell m} \frac{\eta_N}{2} [\hat{a}_k^\dagger, \hat{a}_n] - \delta_{kn} \frac{\eta_N}{2} [\hat{a}_m^\dagger, \hat{a}_\ell] = \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell} \Rightarrow \text{OK!!!}
\end{aligned} \tag{1.33}$$

$$\begin{aligned}
\odot [L_{k\ell}, N_{mn}] &= \left[ L_{k\ell}, \eta_N \frac{1}{2} [\hat{a}_m^\dagger, \hat{a}_n]_{\sigma=\pm} \right] = \frac{\eta_N}{2} [L_{k\ell}, \hat{a}_m^\dagger \hat{a}_n + \sigma \hat{a}_n \hat{a}_m^\dagger] \\
&= \frac{\eta_N}{2} \left( [L_{k\ell}, \hat{a}_m^\dagger] \hat{a}_n + \hat{a}_m^\dagger [L_{k\ell}, \hat{a}_n] + \sigma \left( [L_{k\ell}, \hat{a}_n] \hat{a}_m^\dagger + \hat{a}_n [L_{k\ell}, \hat{a}_m^\dagger] \right) \right) \\
&= \frac{\eta_N}{2} \left( \left[ [L_{k\ell}, \hat{a}_m^\dagger], \hat{a}_n \right]_\sigma + [\hat{a}_m^\dagger, [L_{k\ell}, \hat{a}_n]_\sigma] \right) = \frac{\eta_N}{2} [\hat{a}_m^\dagger, -(\delta_{kn} \hat{a}_\ell^\dagger - \kappa \delta_{\ell n} \hat{a}_k^\dagger)]_\sigma \\
&= -\delta_{kn} \frac{\eta_N}{\eta_L} \frac{\eta_L}{2} [\hat{a}_m^\dagger, \hat{a}_\ell^\dagger]_\sigma + \kappa \delta_{\ell n} \frac{\eta_N}{\eta_L} \frac{\eta_L}{2} [\hat{a}_m^\dagger, \hat{a}_k^\dagger]_\sigma = \frac{\eta_N}{\eta_L} (-\delta_{nk} L_{m\ell} + \kappa \delta_{\ell n} L_{mk}) \Rightarrow \boxed{\eta_L = \eta_N}
\end{aligned} \tag{1.34}$$

$$\begin{aligned}
\odot [L_{k\ell}, M_{mn}] &= \left[ L_{k\ell}, \frac{1}{2}\eta_M [\hat{a}_m, \hat{a}_n]_{\sigma=\pm} \right] = \frac{\eta_M}{2} [L_{k\ell}, \hat{a}_m \hat{a}_n + \sigma \hat{a}_n \hat{a}_m] \\
&= \frac{\eta_M}{2} \left( [L_{k\ell}, \hat{a}_m] \hat{a}_n + \hat{a}_m [L_{k\ell}, \hat{a}_n] + \sigma \left( [L_{k\ell}, \hat{a}_n] \hat{a}_m + \hat{a}_n [L_{k\ell}, \hat{a}_m] \right) \right) \\
&= \frac{\eta_M}{2} \left( \left[ [L_{k\ell}, \hat{a}_m], \hat{a}_n \right]_\sigma + [\hat{a}_m, [L_{k\ell}, \hat{a}_n]_\sigma] \right) \\
&= \frac{\eta_M}{2} \left( \left[ -(\delta_{km} \hat{a}_\ell^\dagger - \kappa \delta_{\ell m} \hat{a}_k^\dagger), \hat{a}_n \right]_\sigma + [\hat{a}_m, -(\delta_{kn} \hat{a}_\ell^\dagger - \kappa \delta_{\ell n} \hat{a}_k^\dagger)]_\sigma \right) \\
&= \frac{\eta_M}{\eta_N} \frac{\eta_N}{2} \left( \left[ -(\delta_{km} \hat{a}_\ell^\dagger - \kappa \delta_{\ell m} \hat{a}_k^\dagger), \hat{a}_n \right]_\sigma + [\hat{a}_m, -(\delta_{kn} \hat{a}_\ell^\dagger - \kappa \delta_{\ell n} \hat{a}_k^\dagger)]_\sigma \right)
\end{aligned}$$

$$\begin{aligned}
&= -\delta_{km} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_\ell^\dagger, \hat{a}_n]_\sigma + \kappa \delta_{\ell m} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_n]_\sigma - \delta_{kn} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_m, \hat{a}_\ell^\dagger]_\sigma + \kappa \delta_{\ell n} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_m, \hat{a}_k^\dagger]_\sigma \\
&= -\delta_{km} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_\ell^\dagger, \hat{a}_n]_\sigma + \kappa \delta_{\ell m} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_n]_\sigma - \delta_{kn} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_\ell^\dagger, \hat{a}_m]_\sigma + \kappa \delta_{\ell n} \frac{\eta_M \eta_N}{\eta_N} \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_m]_\sigma \\
&\stackrel{\sigma=-\kappa}{=} \frac{\eta_M}{\eta_N} (-\delta_{km} N_{\ell n} + \kappa \delta_{\ell m} N_{kn} + \kappa \delta_{kn} N_{\ell m} - \delta_{\ell n} N_{km}) \Rightarrow \boxed{\eta_M = \eta_N} \tag{1.35}
\end{aligned}$$

$$\odot [L_{k\ell}, L_{mn}] = \left[ L_{k\ell}, \frac{1}{2} \eta_L [\hat{a}_m^\dagger, \hat{a}_n^\dagger]_{\sigma=\pm} \right] = 0 \tag{1.36}$$

$$\odot [M_{k\ell}, M_{mn}] = \left[ M_{k\ell}, \frac{1}{2} \eta_L [\hat{a}_m, \hat{a}_n]_{\sigma=\pm} \right] = 0 \tag{1.37}$$

と、(1.28)を再現する。つまり、 $\eta_L = \eta_M = \eta_N = \eta$

$$L_{k\ell} = \frac{1}{2} \eta [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, \quad M_{k\ell} = \frac{1}{2} \eta [\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, \quad N_{k\ell} = \frac{1}{2} \eta [\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \tag{1.38}$$

を得る。更に、(1.21)より  $[N_{\ell m}, \hat{a}_k]$  は

$$\begin{aligned}
[N_{\ell m}, \hat{a}_k] &= \frac{1}{2} \eta [\hat{a}_\ell^\dagger \hat{a}_m + \sigma \hat{a}_m \hat{a}_\ell^\dagger, \hat{a}_k] = \frac{1}{2} \eta [\hat{a}_\ell^\dagger \hat{a}_m \hat{a}_k - \hat{a}_k \hat{a}_\ell^\dagger \hat{a}_m + \sigma (\hat{a}_m \hat{a}_\ell^\dagger \hat{a}_k - \hat{a}_k \hat{a}_m \hat{a}_\ell^\dagger)] \\
&= \frac{1}{2} \eta \left( \hat{a}_\ell^\dagger (\hat{a}_m \hat{a}_k + \kappa \hat{a}_k \hat{a}_m) - (\kappa \hat{a}_\ell^\dagger \hat{a}_k + \hat{a}_k \hat{a}_\ell^\dagger) \hat{a}_m - \left[ \hat{a}_m (\hat{a}_\ell^\dagger \hat{a}_k + \kappa \hat{a}_k \hat{a}_\ell^\dagger) - (\kappa \hat{a}_m \hat{a}_k \hat{a}_\ell^\dagger + \hat{a}_k \hat{a}_m) \hat{a}_\ell^\dagger \right] \right) \tag{1.39} \\
&= \left( \frac{1}{2} \eta \hat{a}_\ell^\dagger [\hat{a}_m, \hat{a}_k]_\kappa - [\hat{a}_k, \hat{a}_\ell^\dagger]_\kappa \hat{a}_m + \hat{a}_m [\hat{a}_\ell^\dagger, \hat{a}_k]_\kappa + [\hat{a}_k, \hat{a}_m]_\kappa \hat{a}_\ell^\dagger \right)
\end{aligned}$$

であるが、**ボーズ粒子とフェルミ粒子**に対して

$$[\hat{a}_\ell, \hat{a}_k^\dagger]_{\kappa=\pm} = \delta_{\ell k}, \quad [\hat{a}_\ell, \hat{a}_k]_{\kappa=\pm} = [\hat{a}_\ell^\dagger, \hat{a}_k^\dagger]_{\kappa=\pm} = 0 \tag{1.40}$$

に注意して

$$[N_{\ell m}, \hat{a}_k] = \frac{1}{2} \eta (-\delta_{k\ell} \hat{a}_m + \hat{a}_m (-\delta_{\ell k})) = -\eta \delta_{k\ell} \hat{a}_m \tag{1.41}$$

より、(1.21)の  $[N_{\ell m}, \hat{a}_k] = -\delta_{\ell k} \hat{a}_m$  は  $\eta = 1$  で満たされる。**パラボーズ粒子とパラフェルミ粒子**に拡張して、最終的に

$$\boxed{L_{k\ell} = \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, \quad M_{k\ell} = \frac{1}{2} [\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, \quad N_{k\ell} = \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (\sigma = -\kappa)} \tag{1.42}$$

を得る。

## 粒子数演算子と生成・消滅演算子

粒子数演算子  $\hat{N}$  により生成演算子  $\hat{a}^\dagger$  と消滅演算子  $\hat{a}$  は

$$[\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \tag{1.43}$$

と定義される。同様に、 $f$ 個のフレーバーに対しては、(1.21)より



$$\begin{aligned} [N_{\ell m}, \hat{a}_k] &= -\delta_{\ell k} \hat{a}_m \Rightarrow [N_{\ell \ell}, \hat{a}_k] = -\delta_{\ell k} \hat{a}_\ell \Rightarrow [N_{kk}, \hat{a}_k] = -\hat{a}_k \\ [N_{\ell m}, \hat{a}_k^\dagger] &= \delta_{mk} \hat{a}_\ell^\dagger \Rightarrow [N_{\ell \ell}, \hat{a}_k^\dagger] = \delta_{\ell k} \hat{a}_\ell^\dagger \Rightarrow [N_{kk}, \hat{a}_k^\dagger] = \hat{a}_k^\dagger \end{aligned} \quad (1.44)$$

であり、 $N_{kk}$  に対しては、(1.28)より

$$N_{m\ell}^\dagger = N_{\ell m} \Rightarrow N_{kk}^\dagger = N_{kk} \Rightarrow N_k \equiv N_{kk} = \frac{1}{2} [\hat{a}_k^\dagger, \hat{a}_k]_{\sigma=\pm} \quad (\sigma = -\kappa) \quad (1.45)$$

$$\begin{aligned} [N_{k\ell}, N_{mn}] &= \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell} \\ \Rightarrow [N_k, N_m] &= \delta_{km} (N_{km} - N_{mk}) = \begin{cases} N_{kk} - N_{kk} = 0 \cdots k = m \\ 0 \cdots k \neq m \end{cases} = 0 \end{aligned} \quad (1.46)$$

と導かれる。まとめると、

$$\begin{cases} [N_k, \hat{a}_k] = -\hat{a}_k \\ [N_k, \hat{a}_k^\dagger] = \hat{a}_k^\dagger \quad (N_k^\dagger = N_k) \\ [N_k, N_m] = 0 \end{cases} \quad (1.47)$$

である。以上から、

- エルミート演算子  $N_k (= N_k^\dagger)$  は、 $[N_k, \hat{a}_k] = -\hat{a}_k$ ,  $[N_k, \hat{a}_k^\dagger] = \hat{a}_k^\dagger$  より粒子数演算子である

ことがわかる。

## パラフェルミ粒子とクリフォード代数: $SO(2f+1)$

フェルミ粒子の場合に、クリフォード代数が構成できた。それには、

$$\gamma_1 = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2i}}, \quad \gamma_2 = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \begin{cases} J_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu], \quad J_{\mu 3} = -J_{3\mu} = \frac{1}{\sqrt{2}} \gamma_\mu \quad (\mu, \nu = 1, 2) \\ [J_{\mu\nu}, \gamma_\rho] = i(\delta_{\mu\rho} \gamma_\nu - \delta_{\nu\rho} \gamma_\mu) \quad (\mu, \nu, \rho = 1, 2, 3) \end{cases} \quad (1.48)$$

を要請する。 $f$ 個のパラフェルミ粒子に用いるため、エルミート演算子

$$\gamma_{2k-1} = \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}}, \quad \gamma_{2k} = \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2i}} \quad (1.49)$$

とする。ここに、

- $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$  は必要としない

ことに注意する。そして、

$$\boxed{J_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu] \quad (\mu, \nu = 1, \dots, 2f)} \quad (1.50)$$

$$\boxed{[J_{\mu\nu}, \gamma_\rho] = i(\delta_{\mu\rho} \gamma_\nu - \delta_{\nu\rho} \gamma_\mu)} \quad (1.51)$$

を設定する。更に

$$\boxed{J_{\mu 2f+1} = -J_{2f+1 \mu} = \frac{1}{\sqrt{2}} \gamma_\mu \quad (\mu, \nu = 1, \dots, 2f)} \quad (1.52)$$

と定義すると、(1.50)の  $[J_{\mu\nu}, \gamma_\rho] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu)$  から

$$\left[ J_{\mu\nu}, \frac{1}{\sqrt{2}} \gamma_\rho \right] = [J_{\mu\nu}, J_{\rho 2f+1}] = i(\delta_{\mu\rho} J_{\nu 2f+1} - \delta_{\nu\rho} J_{\mu 2f+1}) \quad (1.53)$$

を与えるので

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= \begin{cases} i(\delta_{\mu\rho} J_{\nu\sigma} - \delta_{\nu\rho} J_{\mu\sigma}) & (\sigma = 2f+1) \\ i(-\delta_{\mu\sigma} J_{\nu\rho} + \delta_{\nu\sigma} J_{\mu\rho}) & (\rho = 2f+1) \end{cases} \\ &= -i\delta_{\nu\rho} J_{\mu\sigma} + i\delta_{\mu\rho} J_{\nu\sigma} + i\delta_{\nu\sigma} J_{\mu\rho} - i\delta_{\mu\sigma} J_{\nu\rho} \quad (\rho \text{ or } \sigma = 2f+1) \end{aligned} \quad (1.54)$$

である。また、

$$\begin{aligned} \left[ J_{\mu 2f+1}, \frac{1}{\sqrt{2}} \gamma_\rho \right] &= [J_{\mu 2f+1}, J_{\rho 2f+1}] = \left[ \frac{1}{\sqrt{2}} \gamma_\mu, \frac{1}{\sqrt{2}} \gamma_\rho \right] = iJ_{\mu\rho} \\ \left[ J_{2f+1\nu}, \frac{1}{\sqrt{2}} \gamma_\rho \right] &= [J_{2f+1\nu}, J_{\rho 2f+1}] = \left[ -\frac{1}{\sqrt{2}} \gamma_\nu, \frac{1}{\sqrt{2}} \gamma_\rho \right] = -iJ_{\nu\rho} \end{aligned} \quad (1.55)$$

得るので、

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\delta_{\nu\rho} J_{\mu\sigma} + i\delta_{\mu\rho} J_{\nu\sigma} + i\delta_{\nu\sigma} J_{\mu\rho} - i\delta_{\mu\sigma} J_{\nu\rho} \quad (\text{two of } \mu, \nu, \rho, \sigma \text{ are } 2f+1) \quad (1.56)$$

である。従って、(1.54)と(1.56)を考慮すれば  $J_{\mu\nu} (= -J_{\nu\mu})$  ( $\mu, \nu = 1, \dots, 2f+1$ ) は

$$\boxed{[J_{\mu\nu}, J_{\rho\sigma}] = -i\delta_{\nu\rho} J_{\mu\sigma} + i\delta_{\mu\rho} J_{\nu\sigma} + i\delta_{\nu\sigma} J_{\mu\rho} - i\delta_{\mu\sigma} J_{\nu\rho} \quad (\mu, \nu, \rho, \sigma = 1, \dots, 2f+1)} \quad (1.57)$$

を満たし、 $SO(2f+1)$ を記述する。以降、

- $SO(2f+1)$ の記述を明白に導く(1.57)の交換関係と生成消滅演算子の満たす交換関係(1.21)との整合性

を議論する。

(1.52)は生成・消滅演算子で表わすと

$$\begin{aligned} J_{2k-1 2\ell-1} &= \frac{1}{2i} [\gamma_{2k-1}, \gamma_{2\ell-1}] = \frac{1}{2i} \left( \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}} \frac{\hat{a}_\ell - \hat{a}_\ell^\dagger}{\sqrt{2i}} - \frac{\hat{a}_\ell - \hat{a}_\ell^\dagger}{\sqrt{2i}} \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}} \right) \\ &= -\frac{1}{4i} \left( [\hat{a}_k, \hat{a}_\ell] + [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger] - [\hat{a}_k^\dagger, \hat{a}_\ell] + [\hat{a}_\ell^\dagger, \hat{a}_k] \right) = -\frac{1}{2i} (M_{k\ell} + L_{k\ell} - N_{k\ell} + N_{\ell k}) \\ J_{2k-1 2\ell} &= \frac{1}{2i} [\gamma_{2k-1}, \gamma_{2\ell}] = \frac{1}{2i} \left( \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}} \frac{\hat{a}_\ell + \hat{a}_\ell^\dagger}{\sqrt{2}} - \frac{\hat{a}_\ell + \hat{a}_\ell^\dagger}{\sqrt{2}} \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}} \right) \\ &= -\frac{1}{4i} \left( [\hat{a}_k, \hat{a}_\ell] - [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger] - [\hat{a}_k^\dagger, \hat{a}_\ell] - [\hat{a}_\ell^\dagger, \hat{a}_k] \right) = -\frac{1}{2} (M_{k\ell} - L_{k\ell} - N_{k\ell} - N_{\ell k}) \\ J_{2k 2\ell-1} &= \frac{1}{2i} [\gamma_{2k}, \gamma_{2\ell-1}] = \frac{1}{2i} \left( \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} \frac{\hat{a}_\ell - \hat{a}_\ell^\dagger}{\sqrt{2i}} - \frac{\hat{a}_\ell - \hat{a}_\ell^\dagger}{\sqrt{2i}} \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} \right) \\ &= -\frac{1}{4i} \left( [\hat{a}_k, \hat{a}_\ell] - [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger] + [\hat{a}_k^\dagger, \hat{a}_\ell] + [\hat{a}_\ell^\dagger, \hat{a}_k] \right) = -\frac{1}{2} (M_{k\ell} - L_{k\ell} + N_{k\ell} + N_{\ell k}) \end{aligned}$$

$$\begin{aligned}
J_{2k\ 2\ell} &= \frac{1}{2i} [\gamma_{2k}, \gamma_{2\ell}] = \frac{1}{2i} \left( \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} \frac{\hat{a}_\ell + \hat{a}_\ell^\dagger}{\sqrt{2}} - \frac{\hat{a}_\ell + \hat{a}_\ell^\dagger}{\sqrt{2}} \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} \right) \\
&= -\frac{1}{4i} \left( [\hat{a}_k, \hat{a}_\ell] + [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger] + [\hat{a}_k^\dagger, \hat{a}_\ell] - [\hat{a}_\ell^\dagger, \hat{a}_k] \right) = \frac{1}{2i} (M_{k\ell} + L_{k\ell} + N_{k\ell} - N_{\ell k}) \quad (1.58)
\end{aligned}$$

$$J_{2k-1\ 2f+1} = \frac{a_k - a_k^\dagger}{2i}, \quad J_{2k\ 2f+1} = \frac{a_k + a_k^\dagger}{2}$$

を得る。まとめると

$$\begin{aligned}
J_{2k-1\ 2\ell-1} &= -\frac{1}{2i} (M_{k\ell} + L_{k\ell} - N_{k\ell} + N_{\ell k}) \\
J_{2k-1\ 2\ell} &= -\frac{1}{2} (M_{k\ell} - L_{k\ell} - N_{k\ell} - N_{\ell k}) \\
J_{2k\ 2\ell-1} &= -\frac{1}{2} (M_{k\ell} - L_{k\ell} + N_{k\ell} + N_{\ell k}) \\
J_{2k\ 2\ell} &= \frac{1}{2i} (M_{k\ell} + L_{k\ell} + N_{k\ell} - N_{\ell k})
\end{aligned} \quad \& \quad J_{2k-1\ 2f+1} = \frac{\hat{a}_k - \hat{a}_k^\dagger}{2i}, \quad J_{2k\ 2f+1} = \frac{\hat{a}_k + \hat{a}_k^\dagger}{2} \quad (1.59)$$

である。

(1.51)の基本式  $[J_{\mu\nu}, \gamma_\rho] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu)$  が満たされれば、 $SO(2f+1)$ を記述するので、生成・消滅演算子の交換関係から、 $[J_{2k-1\ 2\ell-1}, \gamma_{2m-1}]$ 、 $[J_{2k-1\ 2\ell}, \gamma_{2m-1}]$ 、 $[J_{2k-1\ 2\ell}, \gamma_{2m}]$  や  $[J_{2k\ 2\ell}, \gamma_{2m}]$  を評価し

て、 $[J_{\mu\nu}, \gamma_\rho] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu)$  を再現することを示すと

$$\begin{aligned}
\odot [J_{2k-1\ 2\ell-1}, \gamma_{2m-1}] &= \left[ -\frac{1}{2i} (M_{k\ell} + L_{k\ell} - N_{k\ell} + N_{\ell k}), \frac{\hat{a}_m - \hat{a}_m^\dagger}{\sqrt{2i}} \right] \\
&= \frac{1}{2\sqrt{2}} \left( \left[ \cancel{M_{k\ell}} + L_{k\ell} - N_{k\ell} + N_{\ell k}, \hat{a}_m \right] - \left[ M_{k\ell} + \cancel{L_{k\ell}} - N_{k\ell} + N_{\ell k}, \hat{a}_m^\dagger \right] \right) \\
&\Leftarrow \begin{cases} [N_{k\ell}, \hat{a}_m] = -\delta_{km}\hat{a}_\ell, & [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k^\dagger, & [L_{k\ell}, \hat{a}_m] = -(\delta_{km}\hat{a}_\ell^\dagger - \kappa\delta_{\ell m}\hat{a}_k^\dagger) \\ [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k - \kappa\delta_{km}\hat{a}_\ell, & [L_{\ell m}, \hat{a}_k^\dagger] = [M_{\ell m}, \hat{a}_k] = 0 \end{cases} \quad (1.60)
\end{aligned}$$

$$= \frac{1}{2\sqrt{2}} \left( \begin{aligned} & -(\delta_{km}\hat{a}_\ell^\dagger - \kappa\delta_{\ell m}\hat{a}_k^\dagger) - (-\delta_{km}\hat{a}_\ell) - \delta_{\ell m}\hat{a}_k \\ & -(\delta_{\ell m}\hat{a}_k - \kappa\delta_{km}\hat{a}_\ell - (\delta_{\ell m}\hat{a}_k^\dagger) + \delta_{km}\hat{a}_\ell^\dagger) \end{aligned} \right) \Bigg|_{\kappa=1}$$

$$= i \left( \delta_{km} \frac{\hat{a}_\ell - \hat{a}_\ell^\dagger}{\sqrt{2i}} - \delta_{\ell m} \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2i}} \right) = i(\delta_{km}\gamma_{2\ell-1} - \delta_{\ell m}\gamma_{2k-1})$$

$$[J_{\mu\nu}, \gamma_\rho] = [J_{2k-1\ 2\ell-1}, \gamma_{2m-1}] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu) = i(\delta_{km}\gamma_{2\ell-1} - \delta_{\ell m}\gamma_{2k-1})$$

$$\begin{aligned}
\odot [J_{2k-1\ 2\ell}, \gamma_{2m-1}] &= \left[ -\frac{1}{2} (M_{k\ell} - L_{k\ell} - N_{k\ell} - N_{\ell k}), \frac{\hat{a}_m - \hat{a}_m^\dagger}{\sqrt{2i}} \right] \\
&= -\frac{1}{2\sqrt{2i}} \left( \left[ \cancel{M_{k\ell}} - L_{k\ell} - N_{k\ell} - N_{\ell k}, \hat{a}_m \right] - \left[ M_{k\ell} - \cancel{L_{k\ell}} - N_{k\ell} - N_{\ell k}, \hat{a}_m^\dagger \right] \right)
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k-1\ 2\ell}, \gamma_{2m-1}] &= \left[ -\frac{1}{2}(M_{k\ell} - L_{k\ell} - N_{k\ell} - N_{\ell k}), \frac{\hat{a}_m - \hat{a}_m^\dagger}{\sqrt{2}i} \right] \\
&= -\frac{1}{2\sqrt{2}i} \left( \left[ \cancel{M_{k\ell}} - L_{k\ell} - N_{k\ell} - N_{\ell k}, \hat{a}_m \right] - \left[ M_{k\ell} - \cancel{L_{k\ell}} - N_{k\ell} - N_{\ell k}, \hat{a}_m^\dagger \right] \right) \\
&\Leftrightarrow \begin{cases} [N_{k\ell}, \hat{a}_m] = -\delta_{km}\hat{a}_\ell, [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k^\dagger, [L_{k\ell}, \hat{a}_m] = -(\delta_{km}\hat{a}_\ell^\dagger - \kappa\delta_{\ell m}\hat{a}_k^\dagger) \\ [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k - \kappa\delta_{km}\hat{a}_\ell, [L_{\ell m}, \hat{a}_k^\dagger] = [M_{\ell m}, \hat{a}_k] = 0 \end{cases} \quad (1.61) \\
&= \frac{1}{2\sqrt{2}} i \left( \left( \delta_{km}\hat{a}_\ell^\dagger - \cancel{\kappa\delta_{\ell m}\hat{a}_k^\dagger} \right) - (-\delta_{km}\hat{a}_\ell) + \cancel{\delta_{\ell m}\hat{a}_k} \right) \Bigg|_{\kappa=1} = i\delta_{km} \frac{\hat{a}_\ell^\dagger + \hat{a}_\ell}{\sqrt{2}} = i\delta_{km}\gamma_{2\ell} \\
[J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1\ 2\ell}, \gamma_{2m-1}] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu) = i\delta_{km}\gamma_{2\ell}
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k-1\ 2\ell}, \gamma_{2m}] &= \left[ -\frac{1}{2}(M_{k\ell} - L_{k\ell} - N_{k\ell} - N_{\ell k}), \frac{\hat{a}_m + \hat{a}_m^\dagger}{\sqrt{2}} \right] \\
&= -\frac{1}{2\sqrt{2}} \left( \left[ \cancel{M_{k\ell}} - L_{k\ell} - N_{k\ell} - N_{\ell k}, \hat{a}_m \right] + \left[ M_{k\ell} - \cancel{L_{k\ell}} - N_{k\ell} - N_{\ell k}, \hat{a}_m^\dagger \right] \right) \\
&\Leftrightarrow \begin{cases} [N_{k\ell}, \hat{a}_m] = -\delta_{km}\hat{a}_\ell, [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k^\dagger, [L_{k\ell}, \hat{a}_m] = -(\delta_{km}\hat{a}_\ell^\dagger - \kappa\delta_{\ell m}\hat{a}_k^\dagger) \\ [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k - \kappa\delta_{km}\hat{a}_\ell, [L_{\ell m}, \hat{a}_k^\dagger] = [M_{\ell m}, \hat{a}_k] = 0 \end{cases} \quad (1.62) \\
&= -\frac{1}{2\sqrt{2}} \left( \left( -\left( \cancel{\delta_{km}\hat{a}_\ell^\dagger} - \kappa\delta_{\ell m}\hat{a}_k^\dagger \right) \right) - (-\delta_{km}\hat{a}_\ell) - (-\delta_{\ell m}\hat{a}_k) \right) \Bigg|_{\kappa=1} \\
&= -i\delta_{\ell m} \frac{\hat{a}_k - \hat{a}_k^\dagger}{\sqrt{2}i} = i(-\delta_{\ell m}\gamma_{2k-1}) \\
[J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1\ 2\ell}, \gamma_{2m}] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu) = i(-\delta_{\ell m}\gamma_{2k-1})
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k\ 2\ell}, \gamma_{2m}] &= \left[ \frac{1}{2i}(M_{k\ell} + L_{k\ell} + N_{k\ell} - N_{\ell k}), \frac{\hat{a}_m + \hat{a}_m^\dagger}{\sqrt{2}} \right] \\
&= \frac{1}{2\sqrt{2}i} \left( \left[ \cancel{M_{k\ell}} + L_{k\ell} + N_{k\ell} - N_{\ell k}, \hat{a}_m \right] + \left[ M_{k\ell} + \cancel{L_{k\ell}} + N_{k\ell} - N_{\ell k}, \hat{a}_m^\dagger \right] \right) \\
&\Leftrightarrow \begin{cases} [N_{k\ell}, \hat{a}_m] = -\delta_{km}\hat{a}_\ell, [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k^\dagger, [L_{k\ell}, \hat{a}_m] = -(\delta_{km}\hat{a}_\ell^\dagger - \kappa\delta_{\ell m}\hat{a}_k^\dagger) \\ [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m}\hat{a}_k - \kappa\delta_{km}\hat{a}_\ell, [L_{\ell m}, \hat{a}_k^\dagger] = [M_{\ell m}, \hat{a}_k] = 0 \end{cases} \quad (1.63) \\
&= -\frac{1}{2\sqrt{2}} i \left( \left( -(\delta_{km}\hat{a}_\ell^\dagger - \cancel{\kappa\delta_{\ell m}\hat{a}_k^\dagger}) \right) + (-\delta_{km}\hat{a}_\ell) - (-\delta_{\ell m}\hat{a}_k) \right) \Bigg|_{\kappa=1} \\
&= -i \left( \delta_{\ell m} \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} - \delta_{km} \frac{\hat{a}_\ell + \hat{a}_\ell^\dagger}{\sqrt{2}} \right) = i(\delta_{km}\gamma_{2\ell} - \delta_{\ell m}\gamma_{2k}) \\
[J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1\ 2\ell}, \gamma_{2m}] = i(\delta_{\mu\rho}\gamma_\nu - \delta_{\nu\rho}\gamma_\mu) = i(\delta_{km}\gamma_{2\ell} - \delta_{\ell m}\gamma_{2k})
\end{aligned}$$

により満たされることが分かる。また、 $[J_{2k\ 2f+1}, \gamma_{2m-1}]$ 、 $[J_{2k-1\ 2f+1}, \gamma_{2m-1}]$ 、 $[J_{2k\ 2f+1}, \gamma_{2m}]$  や  $[J_{2k-1\ 2f+1}, \gamma_{2m}]$  は、 $[J_{\mu\nu}, J_{\rho\sigma}] = -i\delta_{\nu\rho}J_{\mu\sigma} + i\delta_{\mu\rho}J_{\nu\sigma} + i\delta_{\nu\sigma}J_{\mu\rho} - i\delta_{\mu\sigma}J_{\nu\rho}$  と同値になる。即ち、

$$\begin{aligned}
\odot [J_{2k\ 2f+1}, \gamma_{2m-1}] &= [J_{2k\ 2f+1}, \sqrt{2}J_{2m-1\ 2f+1}] = \left[ \frac{\hat{a}_k + \hat{a}_k^\dagger}{2}, \frac{\hat{a}_m - \hat{a}_m^\dagger}{\sqrt{2}i} \right] \\
&\Leftarrow L_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, M_{kl} = \frac{1}{2}[\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, N_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (\sigma = -1) \\
&= \frac{1}{2\sqrt{2}i}(2M_{km} + 2N_{km} + 2N_{mk} - 2L_{km}) \Leftarrow J_{2k\ 2\ell-1} = -\frac{1}{2}(M_{kl} - L_{kl} + N_{kl} + N_{\ell k}) \quad (1.64) \\
&= \frac{2}{\sqrt{2}}iJ_{2k\ 2m-1} \Rightarrow [J_{2k\ 2f+1}, J_{2m-1\ 2f+1}] = iJ_{2k\ 2m-1} \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\delta_{\nu\rho}J_{\mu\sigma} + i\delta_{\mu\rho}J_{\nu\sigma} + i\delta_{\nu\sigma}J_{\mu\rho} - i\delta_{\mu\sigma}J_{\nu\rho} = i\delta_{\nu\sigma}J_{2k\ 2m-1} = iJ_{2k\ 2m-1}
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k-1\ 2f+1}, \gamma_{2m-1}] &= [J_{2k-1\ 2f+1}, \sqrt{2}J_{2m-1\ 2f+1}] = \left[ \frac{\hat{a}_k - \hat{a}_k^\dagger}{2i}, \frac{\hat{a}_m - \hat{a}_m^\dagger}{\sqrt{2}i} \right] \\
&\Leftarrow L_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, M_{kl} = \frac{1}{2}[\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, N_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (\sigma = -1) \\
&= -\frac{1}{2\sqrt{2}}(2M_{km} - 2N_{km} + 2N_{mk} + 2L_{km}) \Leftarrow J_{2k-1\ 2\ell-1} = -\frac{1}{2i}(M_{kl} + L_{kl} - N_{kl} + N_{\ell k}) \quad (1.65) \\
&= \sqrt{2}iJ_{2k-1\ 2m-1} \Rightarrow [J_{2k-1\ 2f+1}, J_{2m-1\ 2f+1}] = iJ_{2k-1\ 2m-1} \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\delta_{\nu\rho}J_{\mu\sigma} + i\delta_{\mu\rho}J_{\nu\sigma} + i\delta_{\nu\sigma}J_{\mu\rho} - i\delta_{\mu\sigma}J_{\nu\rho} = i\delta_{\nu\sigma}J_{2k-1\ 2m-1} = iJ_{2k-1\ 2m-1}
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k\ 2f+1}, \gamma_{2m}] &= [J_{2k\ 2f+1}, \sqrt{2}J_{2m\ 2f+1}] = \left[ \frac{\hat{a}_k + \hat{a}_k^\dagger}{2}, \frac{\hat{a}_m + \hat{a}_m^\dagger}{\sqrt{2}} \right] \\
&\Leftarrow L_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, M_{kl} = \frac{1}{2}[\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, N_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (\sigma = -1) \\
&= \frac{1}{2\sqrt{2}}(2M_{km} + 2N_{km} - 2N_{mk} + 2L_{km}) \Leftarrow J_{2k\ 2\ell} = \frac{1}{2i}(M_{kl} + L_{kl} + N_{kl} - N_{\ell k}) \\
&= \sqrt{2}iJ_{2k\ 2m} \Rightarrow [J_{2k\ 2f+1}, J_{2m\ 2f+1}] = iJ_{2k\ 2m} \quad (1.66) \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\delta_{\nu\rho}J_{\mu\sigma} + i\delta_{\mu\rho}J_{\nu\sigma} + i\delta_{\nu\sigma}J_{\mu\rho} - i\delta_{\mu\sigma}J_{\nu\rho} = i\delta_{\nu\sigma}J_{2k\ 2m} = iJ_{2k\ 2m}
\end{aligned}$$

$$\begin{aligned}
\odot [J_{2k-1\ 2f+1}, \gamma_{2m}] &= [J_{2k-1\ 2f+1}, \sqrt{2}J_{2m\ 2f+1}] = \left[ \frac{\hat{a}_k - \hat{a}_k^\dagger}{2i}, \frac{\hat{a}_m + \hat{a}_m^\dagger}{\sqrt{2}} \right] \\
&\Leftarrow L_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell^\dagger]_{\sigma=\pm}, M_{kl} = \frac{1}{2}[\hat{a}_k, \hat{a}_\ell]_{\sigma=\pm}, N_{kl} = \frac{1}{2}[\hat{a}_k^\dagger, \hat{a}_\ell]_{\sigma=\pm} \quad (\sigma = -1) \\
&= -\frac{1}{2\sqrt{2}}i(2M_{km} - 2N_{km} - 2N_{mk} - 2L_{km}) \Leftarrow J_{2k-1\ 2\ell} = -\frac{1}{2}(M_{kl} - L_{kl} - N_{kl} - N_{\ell k}) \quad (1.67) \\
&= \sqrt{2}iJ_{2k-1\ 2m} \Rightarrow [J_{2k-1\ 2f+1}, J_{2m\ 2f+1}] = iJ_{2k-1\ 2m} \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\delta_{\nu\rho}J_{\mu\sigma} + i\delta_{\mu\rho}J_{\nu\sigma} + i\delta_{\nu\sigma}J_{\mu\rho} - i\delta_{\mu\sigma}J_{\nu\rho} = i\delta_{\nu\sigma}J_{2k-1\ 2m} = iJ_{2k-1\ 2m}
\end{aligned}$$

を導く。更に、(1.59)より

$$\begin{cases} M_{k\ell} = \frac{-iJ_{2k-1\ 2\ell-1} + iJ_{2k\ 2\ell} - J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}}{2} \\ L_{k\ell} = \frac{-iJ_{2k-1\ 2\ell-1} + iJ_{2k\ 2\ell} + J_{2k-1\ 2\ell} + J_{2k\ 2\ell-1}}{2} \\ N_{k\ell} = \frac{iJ_{2k-1\ 2\ell-1} + iJ_{2k\ 2\ell} + J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}}{2} \\ N_{\ell k} = \frac{-iJ_{2k-1\ 2\ell-1} - iJ_{2k\ 2\ell} + J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}}{2} \end{cases} \quad (1.68)$$

がわかるので、また、(1.28)は  $J_{\mu\nu}$  ( $\mu, \nu = 1, \dots, 2f+1$ ) の交換関係(1.57)を用いて導くこともでき、

例えば、 $[N_{k\ell}, N_{mn}] = \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell}$  は、

$$\begin{aligned} 4[N_{k\ell}, N_{mn}] &= [iJ_{2k-1\ 2\ell-1} + iJ_{2k\ 2\ell} + J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}, iJ_{2m-1\ 2n-1} + iJ_{2m\ 2n} + J_{2m-1\ 2n} - J_{2m\ 2n-1}] \\ &= i^2 [J_{2k-1\ 2\ell-1} + J_{2k\ 2\ell}, J_{2m-1\ 2n-1} + J_{2m\ 2n}] + [J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}, J_{2m-1\ 2n} - J_{2m\ 2n-1}] \\ &\quad + i [J_{2k-1\ 2\ell-1} + J_{2k\ 2\ell}, J_{2m-1\ 2n} - J_{2m\ 2n-1}] + i [J_{2k-1\ 2\ell} - J_{2k\ 2\ell-1}, J_{2m-1\ 2n-1} + J_{2m\ 2n}] \\ &= -[J_{2k-1\ 2\ell-1}, J_{2m-1\ 2n-1}] - [J_{2k\ 2\ell}, J_{2m\ 2n}] \\ &\quad + [J_{2k-1\ 2\ell}, J_{2m-1\ 2n}] - [J_{2k-1\ 2\ell}, J_{2m\ 2n-1}] - [J_{2k\ 2\ell-1}, J_{2m-1\ 2n}] + [J_{2k\ 2\ell-1}, J_{2m\ 2n-1}] \\ &\quad + i([J_{2k-1\ 2\ell-1}, J_{2m-1\ 2n}] - [J_{2k-1\ 2\ell-1}, J_{2m\ 2n-1}] + [J_{2k\ 2\ell}, J_{2m-1\ 2n}] - [J_{2k\ 2\ell}, J_{2m\ 2n-1}]) \\ &\quad + i([J_{2k-1\ 2\ell}, J_{2m-1\ 2n-1}] + [J_{2k-1\ 2\ell}, J_{2m\ 2n}] - [J_{2k\ 2\ell-1}, J_{2m-1\ 2n-1}] - [J_{2k\ 2\ell-1}, J_{2m\ 2n}]) \\ &\quad \Leftarrow [J_{\mu\nu}, J_{\rho\sigma}] = -i\delta_{\nu\rho} J_{\mu\sigma} + i\delta_{\mu\rho} J_{\nu\sigma} + i\delta_{\nu\sigma} J_{\mu\rho} - i\delta_{\mu\sigma} J_{\nu\rho} \\ &= -(i\delta_{km} J_{2\ell-1\ 2n-1} - i\delta_{kn} J_{2\ell-1\ 2m-1} - i\delta_{\ell m} J_{2k-1\ 2n-1} + i\delta_{\ell n} J_{2k-1\ 2m-1}) \\ &\quad - (i\delta_{km} J_{2\ell\ 2n} - i\delta_{kn} J_{2\ell\ 2m} - i\delta_{\ell m} J_{2k\ 2n} + i\delta_{\ell n} J_{2k\ 2m}) \\ &\quad + i\delta_{km} J_{2\ell\ 2n} + i\delta_{\ell n} J_{2k-1\ 2m-1} - (-i\delta_{\ell m} J_{2k-1\ 2n-1} - i\delta_{kn} J_{2\ell\ 2m}) \\ &\quad - (-i\delta_{kn} J_{2\ell-1\ 2m-1} - i\delta_{\ell m} J_{2k\ 2n}) + i\delta_{km} J_{2\ell-1\ 2n-1} + i\delta_{\ell n} J_{2k\ 2m} \\ &\quad + i \left( (i\delta_{km} J_{2\ell-1\ 2n} - i\delta_{\ell m} J_{2k-1\ 2n}) - (-i\delta_{kn} J_{2\ell-1\ 2m} + i\delta_{\ell n} J_{2k-1\ 2m}) \right) \\ &\quad + i \left( (-i\delta_{kn} J_{2\ell\ 2m-1} + i\delta_{\ell n} J_{2k\ 2m-1}) - (i\delta_{km} J_{2\ell\ 2n-1} - i\delta_{\ell m} J_{2k\ 2n-1}) \right) \\ &\quad + i \left( (i\delta_{km} J_{2\ell\ 2n-1} - i\delta_{kn} J_{2\ell\ 2m-1}) + (-i\delta_{\ell m} J_{2k-1\ 2n} + i\delta_{\ell n} J_{2k-1\ 2m}) \right) \\ &\quad - (-i\delta_{\ell m} J_{2k\ 2n-1} + i\delta_{\ell n} J_{2k\ 2m-1}) - (i\delta_{km} J_{2\ell-1\ 2n} - i\delta_{kn} J_{2\ell-1\ 2m}) \\ &= -(i\delta_{km} \cancel{J_{2\ell-1\ 2n-1}} - i\delta_{kn} \cancel{J_{2\ell-1\ 2m-1}} - i\delta_{\ell m} \cancel{J_{2k-1\ 2n-1}} + i\delta_{\ell n} \cancel{J_{2k-1\ 2m-1}}) \\ &\quad - (i\delta_{km} \cancel{J_{2\ell\ 2n}} - i\delta_{kn} \cancel{J_{2\ell\ 2m}} - i\delta_{\ell m} \cancel{J_{2k\ 2n}} + i\delta_{\ell n} \cancel{J_{2k\ 2m}}) \\ &\quad + i\delta_{km} \cancel{J_{2\ell\ 2n}} + i\delta_{\ell n} J_{2k-1\ 2m-1} - (-i\delta_{\ell m} \cancel{J_{2k-1\ 2n-1}} - i\delta_{kn} \cancel{J_{2\ell\ 2m}}) \\ &\quad - (-i\delta_{kn} \cancel{J_{2\ell-1\ 2m-1}} - i\delta_{\ell m} \cancel{J_{2k\ 2n}}) + i\delta_{km} \cancel{J_{2\ell-1\ 2n-1}} + i\delta_{\ell n} J_{2k\ 2m} \\ &\quad + i \left( (i\delta_{km} \cancel{J_{2\ell-1\ 2n}} - i\delta_{\ell m} \cancel{J_{2k-1\ 2n}}) - (-i\delta_{kn} \cancel{J_{2\ell-1\ 2m}} + i\delta_{\ell n} \cancel{J_{2k-1\ 2m}}) \right) \\ &\quad + i \left( (-i\delta_{kn} \cancel{J_{2\ell\ 2m-1}} + i\delta_{\ell n} \cancel{J_{2k\ 2m-1}}) - (i\delta_{km} \cancel{J_{2\ell\ 2n-1}} - i\delta_{\ell m} \cancel{J_{2k\ 2n-1}}) \right) \end{aligned}$$

$$\begin{aligned}
& + i \left( \begin{aligned} & \left( i\delta_{km} \cancel{J_{2\ell-1, 2n-1}} - i\delta_{kn} \cancel{J_{2\ell, 2m-1}} \right) + \left( -i\delta_{\ell m} \cancel{J_{2k-1, 2n}} + i\delta_{\ell n} J_{2k-1, 2m} \right) \\ & - \left( -i\delta_{\ell m} \cancel{J_{2k, 2n-1}} + i\delta_{\ell n} J_{2k, 2m-1} \right) - \left( i\delta_{km} \cancel{J_{2\ell-1, 2n}} - i\delta_{kn} \cancel{J_{2\ell-1, 2m}} \right) \end{aligned} \right) \\
& = i\delta_{km} \left( -\cancel{J_{2\ell-1, 2n-1}} - \cancel{J_{2\ell, 2n}} + \cancel{J_{2\ell, 2n}} + \cancel{J_{2\ell-1, 2n-1}} + iJ_{2\ell-1, 2n} - iJ_{2\ell, 2n-1} + iJ_{2\ell, 2n-1} - i\cancel{J_{2\ell-1, 2n}} \right) \\
& + i\delta_{kn} \left( J_{2\ell-1, 2m-1} + J_{2\ell, 2m} + J_{2\ell, 2m} + J_{2\ell-1, 2m-1} + iJ_{2\ell-1, 2m} - iJ_{2\ell, 2m-1} - iJ_{2\ell, 2m-1} + iJ_{2\ell-1, 2m} \right) \\
& + i\delta_{\ell m} \left( J_{2k-1, 2n-1} + J_{2k, 2n} + J_{2k-1, 2n-1} + J_{2k, 2n} - iJ_{2k-1, 2n} + iJ_{2k, 2n-1} - iJ_{2k-1, 2n} + iJ_{2k, 2n-1} \right) \\
& + i\delta_{\ell n} \left( -\cancel{J_{2k-1, 2m-1}} - \cancel{J_{2k, 2m}} + \cancel{J_{2k-1, 2m-1}} + \cancel{J_{2k, 2m}} - i\cancel{J_{2k-1, 2m}} + i\cancel{J_{2k, 2m-1}} + i\cancel{J_{2k-1, 2m}} - i\cancel{J_{2k, 2m-1}} \right) \\
& = 2\delta_{kn} \left( iJ_{2\ell-1, 2m-1} + iJ_{2\ell, 2m} - J_{2\ell-1, 2m} + J_{2\ell, 2m-1} \right) + 2\delta_{\ell m} \left( iJ_{2k-1, 2n-1} + iJ_{2k, 2n} + J_{2k-1, 2n} - J_{2k, 2n-1} \right) \\
& \Leftarrow N_{m\ell} = \frac{iJ_{2m-1, 2\ell-1} + iJ_{2m, 2\ell} + J_{2m-1, 2\ell} - J_{2m, 2\ell-1}}{2}, N_{kn} = \frac{iJ_{2k-1, 2n-1} + iJ_{2k, 2n} + J_{2k-1, 2n} - J_{2k, 2n-1}}{2} \\
& = 2\delta_{kn} (-2N_{m\ell}) + 2\delta_{\ell m} (2N_{kn}) = 4(\delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell}) \tag{1.69}
\end{aligned}$$

より、成立することが分かる。

## パラボース粒子と擬クリフォード代数: $Sp(2f, \mathbb{R})$

ボース粒子の場合に、擬クリフォード代数が構成できた。それには、

$$\begin{aligned}
\gamma_1 &= \hat{a}, \gamma_2 = \hat{a}^\dagger \Rightarrow J_{\mu\nu} = \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \}, [J_{\mu\nu}, \gamma_\rho] = g_{\mu\rho} \gamma_\nu + g_{\nu\rho} \gamma_\mu \quad (\mu, \nu = 1, 2) \\
g_{\mu\nu} &= \begin{cases} 1 \cdots (\mu, \nu) = (1, 2), & -1 \cdots (\mu, \nu) = (2, 1) \\ 0 \cdots (\mu, \nu) = (1, 1), (2, 2) \end{cases} = -g_{\nu\mu} \tag{1.70}
\end{aligned}$$

を要請する。 $f$ 個のパラボース粒子に用いるため、エルミート演算子

$$\gamma_{2k-1} = \hat{a}_k, \gamma_{2k} = \hat{a}_k^\dagger \quad (k = 1, \dots, f) \tag{1.71}$$

とする。ここに、

- $[ \gamma_\mu, \gamma_\nu ] = g_{\mu\nu}$  は必要としない

ことに注意する。そして、

$$J_{\mu\nu} = \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \} \quad (\mu, \nu = 1, \dots, 2f) \tag{1.72}$$

を構成し

$$\boxed{ [ J_{\mu\nu}, \gamma_\rho ] = g_{\mu\rho} \gamma_\nu + g_{\nu\rho} \gamma_\mu \quad (\mu, \nu = 1, \dots, 2f) } \tag{1.73}$$

を設定する。ここに

$$\boxed{ g_{\mu\nu} = \begin{cases} 1 \cdots (\mu, \nu) = (\text{odd}, \text{even}), & -1 \cdots (\mu, \nu) = (\text{even}, \text{odd}) \\ 0 \cdots (\mu, \nu) = (\text{even}, \text{even}), (\text{odd}, \text{odd}) \end{cases} = -g_{\nu\mu} } \tag{1.74}$$

である。そのとき、 $J_{\mu\nu} (= J_{\nu\mu})$  は

$$\boxed{[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}J_{\nu\sigma} + g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} + g_{\nu\sigma}J_{\mu\rho} \text{ with } J_{\mu\nu} = J_{\nu\mu} \ (\mu, \nu, \rho, \sigma = 1, \dots, 2f)} \quad (1.75)$$

を満たし、 $Sp(2f, \mathbf{R})$ を記述する。以降、前節と同様に

- $Sp(2f, \mathbf{R})$ の記述を明白に導く(1.75)の交換関係と生成消滅演算子の満たす交換関係(1.21)との整合性

を議論する。

(1.72)は生成消滅演算子で表わすと

$$\boxed{\begin{aligned} J_{2k-1 \ 2\ell-1} &= \frac{1}{2} \{\gamma_{2k-1}, \gamma_{2\ell-1}\} = \frac{1}{2} \{\hat{a}_k, \hat{a}_\ell\} = M_{k\ell} \\ J_{2k-1 \ 2\ell} &= \frac{1}{2} \{\gamma_{2k-1}, \gamma_{2\ell}\} = \frac{1}{2} \{\hat{a}_k, \hat{a}_\ell^\dagger\} = N_{k\ell} = N_{k\ell}^\dagger \\ J_{2k \ 2\ell-1} &= \frac{1}{2} \{\gamma_{2k}, \gamma_{2\ell-1}\} = \frac{1}{2} \{\hat{a}_k^\dagger, \hat{a}_\ell\} = N_{k\ell} \\ J_{2k \ 2\ell} &= \frac{1}{2} \{\gamma_{2k}, \gamma_{2\ell}\} = \frac{1}{2} \{\hat{a}_k^\dagger, \hat{a}_\ell^\dagger\} = L_{k\ell} \end{aligned}} \quad (1.76)$$

を得る。ここに、 $J_{2k-1 \ 2\ell} = J_{2\ell \ 2k-1}$ を満たす。(1.73)の基本式 $[J_{\mu\nu}, \gamma_\rho] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu$ が満たされれば、 $Sp(2f, \mathbf{R})$ を記述する。実際、 $[J_{2k-1 \ 2\ell-1}, \gamma_{2m-1}]$ 、 $[J_{2k-1 \ 2\ell}, \gamma_{2m-1}]$ 、 $[J_{2k-1 \ 2\ell}, \gamma_{2m}]$ や $[J_{2k \ 2\ell}, \gamma_{2m}]$ に $M_{k\ell}$ 、 $N_{k\ell}$ 、 $L_{k\ell}$ の交換関係(1.21)を用いると $[J_{\mu\nu}, \gamma_\rho] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu$ を再現する。ここで、 $g_{\text{odd odd}} = g_{\text{even even}} = 0$ 、 $g_{\text{odd even}} = -g_{\text{even odd}} = 1$ に注意して

$$\begin{aligned} \odot [J_{2k-1 \ 2\ell-1}, \gamma_{2m-1}] &= [M_{k\ell}, \hat{a}_m] = 0 \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1 \ 2\ell-1}, \gamma_{2m-1}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k-1 \ 2m-1}\gamma_{2\ell-1} + g_{2\ell-1 \ 2m-1}\gamma_{2k-1} = 0 \\ \odot [J_{2k-1 \ 2\ell-1}, \gamma_{2m}] &= [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m} \hat{a}_k + \delta_{km} \hat{a}_\ell \leftarrow [M_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m} \hat{a}_k - \kappa \delta_{km} \hat{a}_\ell \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1 \ 2\ell-1}, \gamma_{2m}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k-1 \ 2m}\gamma_{2\ell-1} + g_{2\ell-1 \ 2m}\gamma_{2k-1} = \delta_{km} \hat{a}_\ell + \delta_{\ell m} \hat{a}_k \\ \odot [J_{2k-1 \ 2\ell}, \gamma_{2m-1}] &= [N_{k\ell}, \hat{a}_m] = -\delta_{\ell m} \hat{a}_k \leftarrow [N_{k\ell}, \hat{a}_m] = -\delta_{km} \hat{a}_\ell \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1 \ 2\ell}, \gamma_{2m-1}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k-1 \ 2m-1}\gamma_{2\ell} + g_{2\ell-1 \ 2m-1}\gamma_{2k-1} = -\delta_{\ell m} \hat{a}_k \\ \odot [J_{2k-1 \ 2\ell}, \gamma_{2m}] &= [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{km} \hat{a}_\ell^\dagger \leftarrow [N_{k\ell}, \hat{a}_m^\dagger] = \delta_{\ell m} \hat{a}_k^\dagger \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k-1 \ 2\ell}, \gamma_{2m}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k-1 \ 2m}\gamma_{2\ell} + g_{2\ell \ 2m}\gamma_{2k-1} = \delta_{km} \hat{a}_\ell \\ \odot [J_{2k \ 2\ell}, \gamma_{2m-1}] &= [L_{k\ell}, \hat{a}_m] = -(\delta_{km} \hat{a}_\ell^\dagger + \delta_{\ell m} \hat{a}_k^\dagger) \leftarrow [L_{k\ell}, \hat{a}_m] = -(\delta_{km} \hat{a}_\ell^\dagger - \kappa \delta_{\ell m} \hat{a}_k^\dagger) \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k \ 2\ell}, \gamma_{2m-1}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k \ 2m-1}\gamma_{2\ell} + g_{2\ell \ 2m-1}\gamma_{2k} = -\delta_{km} \hat{a}_\ell^\dagger - \delta_{\ell m} \hat{a}_k^\dagger \\ \odot [J_{2k \ 2\ell}, \gamma_{2m}] &= [L_{k\ell}, \hat{a}_m^\dagger] = 0 \\ [J_{\mu\nu}, \gamma_\rho] &= [J_{2k \ 2\ell}, \gamma_{2m}] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu = g_{2k \ 2m}\gamma_{2\ell} + g_{2\ell \ 2m}\gamma_{2k} = 0 \end{aligned} \quad (1.77)$$

により $[J_{\mu\nu}, \gamma_\rho] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu$ が満たされることが分かる。また、(1.28)は $J_{\mu\nu} (\mu, \nu = 1, \dots, 2f)$ の交換関係(1.75)を用いて導くこともでき、例えば、



$$\begin{aligned}
[N_{k\ell}, N_{mn}] &= [J_{2k\ 2\ell-1}, J_{2m\ 2n-1}] = [J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}J_{\nu\sigma} + g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} + g_{\nu\sigma}J_{\mu\rho} \\
&= g_{2k\ 2m}J_{2\ell-1\ 2n-1} + g_{2\ell-1\ 2m}J_{2k\ 2n-1} + g_{2k\ 2n-1}J_{2\ell-1\ 2m} + g_{2\ell-1\ 2n-1}J_{2k\ 2m} \\
&= \delta_{\ell m}N_{kn} - \delta_{kn}N_{m\ell}
\end{aligned} \tag{1.78}$$

より、 $[N_{k\ell}, N_{mn}] = \delta_{\ell m}N_{kn} - \delta_{kn}N_{m\ell}$  が成立することが分かる。

## $Sp(2f, \mathbf{R})$ の拡張

更に、 $J_\mu$ として

$$\boxed{J_{2k-1} = \frac{\gamma_{2k-1}}{\sqrt{2}} \left( = \frac{a_k}{\sqrt{2}} \right), J_{2k} = \frac{\gamma_{2k}}{\sqrt{2}} \left( = \frac{a_k^\dagger}{\sqrt{2}} \right)} \tag{1.79}$$

を定義すると、 $[J_{\mu\nu}, \gamma_\rho] = g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu$  より

$$\boxed{[J_{\mu\nu}, J_\rho] = g_{\mu\rho}J_\nu + g_{\nu\rho}J_\mu} \tag{1.80}$$

を与える。一方、(1.76)より

$$\left\{ \frac{\gamma_\mu}{\sqrt{2}}, \frac{\gamma_\nu}{\sqrt{2}} \right\} = J_{\mu\nu} \Rightarrow \{J_\mu, J_\nu\} = J_{\mu\nu} \tag{1.81}$$

を得る。(1.81)を交換関係で定義するには、新たに $S_\mu$ としてしてグラスマン数 $\theta$ と共に

$$S_\mu(\theta) = \theta J_\mu \tag{1.82}$$

とすると、(1.80)と(1.81)は

$$\begin{cases} [J_{\mu\nu}, S_\rho(\theta)] = g_{\mu\rho}S_\nu(\theta) + g_{\nu\rho}S_\mu(\theta) \\ [S_\mu(\theta), S_\nu(\theta')] = \theta\theta'J_{\mu\nu} \text{ with } \{\theta, \theta'\} = 0 \end{cases} \quad (\mu, \nu = 1, \dots, 2f) \tag{1.83}$$

とまとめられる。実際、

$$\begin{aligned}
[S_\mu(\theta), S_\nu(\theta')] &= [\theta J_{\mu\ 2f+1}, \theta' J_{\nu\ 2f+1}] = \theta J_{\mu\ 2f+1} \theta' J_{\nu\ 2f+1} - \theta' J_{\nu\ 2f+1} \theta J_{\mu\ 2f+1} \\
&= \theta\theta' J_{\mu\ 2f+1} J_{\nu\ 2f+1} - (-\theta\theta') J_{\nu\ 2f+1} J_{\mu\ 2f+1} = \theta\theta' \{J_{\mu\ 2f+1}, J_{\nu\ 2f+1}\} = \theta\theta' J_{\mu\nu}
\end{aligned} \tag{1.84}$$

より、(1.81)を導く。また、 $[S_\mu(\theta), S_\nu(\theta')] = \theta\theta'J_{\mu\nu}$  及び  $[S_\nu(\theta'), S_\mu(\theta)] = \theta'\theta J_{\nu\mu} = -\theta\theta'J_{\nu\mu}$  より、

$J_{\nu\mu} = J_{\mu\nu}$  を導く。以上より、 $Sp(2f, \mathbf{R})$ の生成子は $(J_{\mu\nu}, S_\mu(\theta))$ に拡張され

$$\boxed{
\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= g_{\mu\rho}J_{\nu\sigma} + g_{\nu\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\rho} + g_{\nu\sigma}J_{\mu\rho} \text{ with } J_{\mu\nu} = J_{\nu\mu} \quad (\mu, \nu, \rho, \sigma = 1, \dots, 2f) \\
[J_{\mu\nu}, S_\rho(\theta)] &= g_{\mu\rho}S_\nu(\theta) + g_{\nu\rho}S_\mu(\theta) \text{ for } S_\mu = \theta J_\mu \\
[S_\mu(\theta), S_\nu(\theta')] &= \theta\theta'J_{\mu\nu} \text{ with } \{\theta, \theta'\} = 0
\end{aligned}
} \tag{1.85}$$

を満たすことがわかる。