

第三章 量子統計

粒子数を $n(\mathbf{p})$ とすると、その分布は

$$\frac{dn(\mathbf{p})}{d\mathbf{p}} \equiv f(\mathbf{p}) = f(p) = \frac{g}{(2\pi\hbar)^3} \frac{1}{e^{(E-\mu)/kT} + \theta} \quad \text{with } E = \sqrt{p^2 c^2 + m^2 c^4} \leftarrow \begin{cases} \theta = 1 \text{ for Fermi-Dirac} \\ \theta = -1 \text{ for Bose-Einstein} \end{cases}$$

で決められる。また、 g はスピンからの自由度を表し

$$g = \begin{cases} 2s+1 & \text{for massive spin } s \\ 2 & \text{for massless spin } s (\neq 0) \\ 1 & \text{for massless/Majorana } \nu_L \end{cases}$$

である。これから、

$$n(\mathbf{p}) = \int dN(\mathbf{p}) = \int d\mathbf{p} f(p) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{g}{e^{(E-\mu)/kT} + \theta} = \int \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \frac{g}{e^{(E-\mu)/kT} + \theta} = \frac{g}{2\pi^2 \hbar^3} \int \frac{p^2 dp}{e^{(E-\mu)/kT} + \theta}$$

である。さらに積分を

$$z = \frac{E}{kT} = \frac{\sqrt{p^2 c^2 + m^2 c^4}}{kT}$$

で書き換え、

$$\int \frac{p^2 dp}{e^{(E-\mu)/kT} + \theta} = \int \frac{p^2 dp}{e^{z-\mu/kT} + \theta}$$

を評価する。そこで、

$$z = \frac{\sqrt{p^2 c^2 + m^2 c^4}}{kT} \Rightarrow (kT)^2 z^2 = p^2 c^2 + m^2 c^4 \Rightarrow (kT)^2 2z dz = 2c^2 p dp \Rightarrow (kT)^2 z dz = c^2 p dp$$

$$\Rightarrow (kT)^2 z p dz = c^2 p^2 dp \Rightarrow p^2 dp = \frac{(kT)^2}{c^2} z p dz = \frac{(kT)^2}{c^2} z \frac{\sqrt{(kT)^2 z^2 - m^2 c^4}}{c} dz$$

$$\Rightarrow p^2 dp = \left(\frac{kT}{c}\right)^3 z \sqrt{z^2 - \left(\frac{mc^2}{kT}\right)^2} dz$$

$$p: 0 \rightarrow \infty \Rightarrow z: \frac{mc^2}{kT} \rightarrow \infty = x \rightarrow \infty \left(x = \frac{mc^2}{kT} \right)$$

を用いて、

$$\begin{aligned} n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) &\equiv \int N(\mathbf{p}) d\mathbf{p} = \frac{g}{2\pi^2 \hbar^3} \int \frac{p^2 dp}{e^{z-\mu/kT} + \theta} = \frac{g}{2\pi^2 \hbar^3} \int_x^\infty \frac{1}{e^{z-\mu/kT} + \theta} \left(\frac{kT}{c}\right)^3 z \sqrt{z^2 - \left(\frac{mc^2}{kT}\right)^2} dz \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^\infty z \sqrt{z^2 - x^2} (e^{z-\mu/kT} + \theta)^{-1} dz \leftarrow x = \frac{mc^2}{kT} \end{aligned}$$

を得る。従って、

$$n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^\infty z \sqrt{z^2 - x^2} (e^{z-\mu/kT} + \theta)^{-1} dz \quad \left(x = \frac{mc^2}{kT}\right)$$

である。この計算は、【Appendix 1:種々の特殊関数の公式—ベッセル関数】を用いる。そこで、

$$\begin{aligned} (e^{z-\mu/kT} + \theta)^{-1} &= \frac{e^{-z+\mu/kT}}{1+\theta e^{-z+\mu/kT}} = e^{-z+\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{-z+\mu/kT})^k = e^{\mu/kT} \sum_{k=0}^{\infty} e^{-z} (e^{-z})^k (-\theta e^{\mu/kT})^k \\ &= e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k e^{-z(k+1)} \end{aligned}$$

より、

$$\begin{aligned} n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^\infty z \sqrt{z^2 - x^2} (e^{z-\mu/kT} + \theta)^{-1} dz \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^\infty z \sqrt{z^2 - x^2} e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k e^{-z(k+1)} dz \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k \int_x^\infty z \sqrt{z^2 - x^2} e^{-z(k+1)} dz \end{aligned}$$

Appendix 1:種々の特殊関数の公式—ベッセル関数

$$\begin{aligned} &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k \left[2^{\nu-\frac{1}{2}} (\sqrt{\pi})^{-1} \alpha^{\frac{1}{2}-\nu} x^{\nu+\frac{1}{2}} \Gamma(\nu) K_{\nu+\frac{1}{2}}(\alpha x) \right] \Bigg|_{\nu=\frac{3}{2}, \alpha=k+1} \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k \left[2^{\frac{3}{2}-\frac{1}{2}} (\sqrt{\pi})^{-1} (k+1)^{\frac{1}{2}-\frac{3}{2}} x^{\frac{3}{2}+\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) K_{\frac{3}{2}+\frac{1}{2}}((k+1)x) \right] \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} (-\theta e^{\mu/kT})^k \left[2 \frac{1}{\sqrt{\pi}} \frac{1}{k+1} x^2 \frac{1}{2} \sqrt{\pi} K_2((k+1)x) \right] \\ &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} \frac{(-\theta e^{\mu/kT})^k}{k+1} x^2 K_2((k+1)x) \end{aligned}$$

従って、

$$n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} \frac{(-\theta e^{\mu/kT})^k}{k+1} x^2 K_2((k+1)x)$$

を得る。

第1節 $mc^2 \gg kT$ ($x \gg 0$)

このとき、 $x \gg 1$ なので、

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{4\nu^2 - 1}{8x}\right) \quad (x \gg 1) \quad \leftarrow x = \frac{mc^2}{kT}$$

と近似できる。主に $k=0$ が効くので、

$$\begin{aligned}
 n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \sum_{k=0}^{\infty} \frac{(-\theta e^{\mu/kT})^k}{k+1} x^2 K_2((k+1)x) \\
 &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} x^2 K_2(x) \sim \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} x^2 \sqrt{\frac{\pi}{2x}} e^{-x} \\
 &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \left(\frac{mc^2}{kT}\right)^2 \sqrt{\frac{\pi}{2\frac{mc^2}{kT}}} e^{-\frac{mc^2}{kT}} = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{-(mc^2-\mu)/kT} \left(\frac{mc^2}{kT}\right)^{3/2} \sqrt{\frac{\pi}{2}} = g \left(\frac{mc^2 kT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT}
 \end{aligned}$$

従って、

$$n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) = g \left(\frac{mc^2 kT}{2\pi\hbar^2}\right)^{3/2} e^{-(mc^2-\mu)/kT} \text{ for } kT \ll mc^2$$

を得る。

第2節 $mc^2 \ll kT$ ($x \approx 0$) with $\mu=0$

もとの表式から

$$\begin{aligned}
 n\left(T, \frac{mc^2}{kT}, 0\right) &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^{\infty} z \sqrt{z^2 - x^2} (e^z + \theta)^{-1} dz \quad \left(x = \frac{mc^2}{kT}\right) \\
 &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^{\infty} z^2 \sqrt{1 - \left(\frac{x}{z}\right)^2} (e^z + \theta)^{-1} dz = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^{\infty} z^2 \left(1 - \frac{1}{2}\left(\frac{x}{z}\right)^2 + \frac{1}{8}\left(\frac{x}{z}\right)^4 + \dots\right) (e^z + \theta)^{-1} dz \\
 &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\int_x^{\infty} \frac{z^2}{e^z + \theta} dz - \frac{1}{2} x^2 \int_x^{\infty} \frac{1}{e^z + \theta} dz + \dots\right)
 \end{aligned}$$

ここで、第2項は

$$\int_x^{\infty} \frac{1}{e^z + \theta} dz \stackrel{Z=e^z}{=} \int_{e^x}^{\infty} \frac{1}{Z + \theta} \frac{dZ}{Z} = \begin{cases} \int_{e^x}^{\infty} \left(\frac{1}{Z} - \frac{1}{Z+1}\right) dZ = \ln \frac{Z}{Z+1} \Big|_{e^x}^{\infty} = -\ln \frac{e^x}{e^x+1} & \text{for } \theta = 1 \\ \int_{e^x}^{\infty} \left(\frac{1}{Z-1} - \frac{1}{Z}\right) dZ = \ln \frac{Z-1}{Z} \Big|_{e^x}^{\infty} = -\ln \frac{e^x-1}{e^x} & \text{for } \theta = -1 \\ \int_{e^x}^{\infty} \frac{1}{Z^2} dZ = -Z \Big|_{e^x}^{\infty} = e^x & \text{for } \theta = 0 \end{cases}$$

従って、

$$\begin{aligned}
 (\theta = 1) &= \int_x^{\infty} \frac{1}{e^z + 1} dz = \ln \frac{e^x + 1}{e^x} = \ln \frac{2 + \dots}{1 + \dots} = \ln 2 + \dots \\
 (\theta = -1) &= \int_x^{\infty} \frac{1}{e^z - 1} dz = -\ln \frac{e^x - 1}{e^x} = -\ln \frac{1 + x + \dots - 1}{1 + \dots} + \dots = -\ln x + \dots \\
 (\theta = 0) &= \int_x^{\infty} \frac{1}{e^z} dz = e^x = 1 + \dots
 \end{aligned}$$

また、第1項は、

$$\begin{aligned}
\int_x^\infty \frac{z^2}{e^z + \theta} dz &= \int_0^\infty \frac{z^2}{e^z + \theta} dz + x \cdot \frac{d}{dx} \left(\int_x^\infty \frac{z^2}{e^z + \theta} dz \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \frac{d^2}{dx^2} \left(\int_x^\infty \frac{z^2}{e^z + \theta} dz \right) \Big|_{x=0} + \dots \\
&= \int_0^\infty \frac{z^2}{e^z + \theta} dz + x \cdot \left(-\frac{x^2}{e^x + \theta} \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \frac{d}{dx} \left(-\frac{x^2}{e^x + \theta} \right) \Big|_{x=0} + \dots \\
&= \int_0^\infty \frac{z^2}{e^z + \theta} dz + x \cdot \left(-\frac{x^2}{e^x + \theta} \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \left(\frac{x^2 e^x}{(e^x + \theta)^2} - \frac{2x}{e^x + \theta} \right) \Big|_{x=0} + \dots
\end{aligned}$$

さらに、

$$\begin{aligned}
(\theta = 1) &= \int_x^\infty \frac{z^2}{e^z + 1} dz = \int_0^\infty \frac{z^2}{e^z + 1} dz + x \cdot \left(-\frac{x^2}{e^x + 1} \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \left(\frac{x^2 e^x}{(e^x + 1)^2} - \frac{2x}{e^x + 1} \right) \Big|_{x=0} + \dots \\
&= \int_0^\infty \frac{z^2}{e^z + 1} dz + \dots
\end{aligned}$$

$$\begin{aligned}
(\theta = -1) &= \int_x^\infty \frac{z^2}{e^z - 1} dz = \int_0^\infty \frac{z^2}{e^z - 1} dz + x \cdot \left(-\frac{x^2}{e^x - 1} \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \left(\frac{x^2 e^x}{(e^x - 1)^2} - \frac{2x}{e^x - 1} \right) \Big|_{x=0} + \dots \\
&= \int_0^\infty \frac{z^2}{e^z - 1} dz + x \cdot \left(-\frac{x^2}{x} \right) \Big|_{x=0} + \frac{x^2}{2!} \cdot \left(\frac{x^2}{x^2} - \frac{2x}{x} \right) \Big|_{x=0} + \dots \\
&= \int_0^\infty \frac{z^2}{e^z - 1} dz + \frac{x^2}{2!} (1 - 2) + \dots = \int_0^\infty \frac{z^2}{e^z - 1} dz - \frac{x^2}{2!} + \dots
\end{aligned}$$

$$\begin{aligned}
(\theta = 0) &= \int_x^\infty \frac{z^2}{e^z} dz = [2(-1 - z) - z^2] e^{-z} \Big|_x^\infty = [2(1 + x) + x^2] e^{-x} \\
&= [2(1 + x) + x^2] \left(1 - x + \frac{x^2}{2!} \right) + \dots = 2(1 - x^2) + x^2 + 2 \frac{x^2}{2} + \dots = 2 + \dots
\end{aligned}$$

なので、

$$n \left(T, \frac{mc^2}{kT}, 0 \right) = \begin{cases} (\theta = 1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar} \right)^3 \left(\int_0^\infty \frac{z^2}{e^z + 1} dz - \frac{1}{2} x^2 \ln 2 + \dots \right) \\ (\theta = -1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar} \right)^3 \left(\int_0^\infty \frac{z^2}{e^z - 1} dz - \frac{x^2}{2!} + \frac{1}{2} x^2 \ln x + \dots \right) \leftarrow \begin{cases} \theta = 1 \text{ for Fermi-Dirac} \\ \theta = -1 \text{ for Bose-Einstein} \end{cases} \\ (\theta = 0) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar} \right)^3 \left(2 - \frac{x^2}{2!} + \dots \right) \end{cases}$$

残っている積分は、【Appendix 1: 種々の特殊関数の公式—ツェータ関数】を用い、

$$\int_0^\infty \frac{z^2}{e^z + \theta} dz = \begin{cases} (\theta = 1) = \frac{1}{1^3} (1 - 2^{1-3}) \Gamma(3) \zeta(3) = \frac{3}{4} \Gamma(3) \zeta(3) \stackrel{\Gamma(3)=2\Gamma(2)=2 \cdot 1\Gamma(1)=2}{=} \frac{3}{2} \zeta(3) \\ (\theta = -1) = \frac{1}{1^3} \Gamma(3) \zeta(3) = \Gamma(3) \zeta(3) \stackrel{\Gamma(3)=2\Gamma(2)=2 \cdot 1\Gamma(1)=2}{=} 2 \zeta(3) \end{cases}$$

及び、

$$n\left(T, \frac{mc^2}{kT}, 0\right) = \begin{cases} (\theta=1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\int_0^\infty \frac{z^2}{e^z + 1} dz - \frac{1}{2} x^2 \ln 2 + \dots \right) \\ (\theta=-1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\int_0^\infty \frac{z^2}{e^z - 1} dz - \frac{x^2}{2!} + \frac{1}{2} x^2 \ln x + \dots \right) \end{cases} \leftarrow x = \frac{mc^2}{kT}$$

$$(\theta=1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\int_0^\infty \frac{z^2}{e^z + 1} dz - \frac{1}{2} x^2 \ln 2 + \dots \right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{3}{2} \zeta(3) - \frac{1}{2} x^2 \ln 2 + \dots \right)$$

$$= \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{3}{4} \zeta(3) - \frac{1}{4} x^2 \ln 2 + \dots \right)$$

$$(\theta=-1) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\int_0^\infty \frac{z^2}{e^z - 1} dz - \frac{x^2}{2!} + \frac{1}{2} x^2 \ln x + \dots \right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(2\zeta(3) - \frac{x^2}{2!} + \frac{1}{2} x^2 \ln x + \dots \right)$$

$$= \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\zeta(3) - \frac{x^2}{4} + \frac{1}{4} x^2 \ln x + \dots \right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\zeta(3) + \frac{1}{4} x^2 (\ln x - 1) + \dots \right)$$

以上から、

$$n\left(T, \frac{mc^2}{kT}, 0\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{3}{4} \zeta(3) - \frac{1}{4} \left(\frac{mc^2}{kT}\right)^2 \ln 2 + \dots \right) \quad (\theta=1 \text{ for Fermi-Dirac})$$

$$n\left(T, \frac{mc^2}{kT}, 0\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\zeta(3) + \frac{1}{4} \left(\frac{mc^2}{kT}\right)^2 \left(\ln \frac{mc^2}{kT} - 1 \right) + \dots \right) \quad (\theta=-1 \text{ for Bose-Einstein})$$

$$n\left(T, \frac{mc^2}{kT}, 0\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(1 - \frac{x^2}{4} + \dots \right) \quad (\theta=0 \text{ for Maxwell-Boltzmann})$$

を得る。¹

第3節 $m=0$

もとの表式

$$n\left(T, \frac{mc^2}{kT}, \frac{\mu}{kT}\right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_x^\infty z \sqrt{z^2 - x^2} \left(e^{z - \mu/kT} + \theta \right)^{-1} dz \quad \left(x = \frac{mc^2}{kT} \right)$$

より、

$$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \int_0^\infty z^2 \left(e^{z - \mu/kT} + \theta \right)^{-1} dz = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \int_0^\infty z^2 \left(e^z + \theta e^{\mu/kT} \right)^{-1} dz$$

なので、

¹ $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202$

$$\begin{aligned}
n\left(T, 0, \frac{\mu}{kT}\right) &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \int_0^\infty z^2 (e^z + \theta e^{\mu/kT})^{-1} dz = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \frac{Li_3(-\theta e^{\mu/kT}) \Gamma(3)}{-\theta e^{\mu/kT}} \\
&= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 \frac{Li_3(-\theta e^{\mu/kT}) 2}{-\theta} = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \frac{Li_3(-\theta e^{\mu/kT})}{-\theta} \quad (\theta = \pm 1) \\
n\left(T, 0, \frac{\mu}{kT}\right) &= \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \int_0^\infty z^2 (e^z + 0)^{-1} dz = \frac{g}{2\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \Gamma(3) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \quad (\theta = 0)
\end{aligned}$$

とわかる。従って、

$n\left(T, 0, \frac{\mu}{kT}\right) = -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(-e^{\mu/kT}) \quad (\theta = 1 \text{ for Fermi-Dirac})$
$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(e^{\mu/kT}) \quad (\theta = -1 \text{ for Bose-Einstein})$
$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 e^{\mu/kT} \quad (\theta = 0 \text{ for Maxwell-Boltzmann})$

【 $\mu/kT \ll 1$ のとき】

また、【Appendix 1: 種々の特殊関数の公式—ツェータ関数】の

$$\begin{aligned}
Li_3(1-x) &= \zeta(3) - x\zeta(2) - \frac{1}{4}x^2(2\ln x + 2\zeta(2) - 3) + \dots \\
Li_3(-1+x) &= \left(\frac{1}{4} - 1\right)\zeta(3) + x\left(1 - \frac{1}{2}\right)\zeta(2) + \dots
\end{aligned}$$

を用いると、

$$\frac{\mu}{kT} \ll 1$$

のとき、

$$\begin{aligned}
(\theta = 1) &= n\left(T, \frac{mc^2}{kT}, 0\right) = -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(-e^{\mu/kT}) \approx -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3\left(-1 - \frac{\mu}{kT}\right) \\
&= -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left[\left(\frac{1}{4} - 1\right)\zeta(3) - \frac{\mu}{kT} \left(1 - \frac{1}{2}\right)\zeta(2) + \dots \right] \\
&= -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(-\frac{3}{4}\zeta(3) - \frac{\mu}{kT} \frac{1}{2}\zeta(2) + \dots \right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{3}{4}\zeta(3) + \frac{\mu}{2kT}\zeta(2) + \dots \right) \\
(\theta = -1) &= n\left(T, \frac{mc^2}{kT}, 0\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(e^{\mu/kT}) \approx \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3\left(1 + \frac{\mu}{kT}\right) \\
&= \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\zeta(3) + \frac{\mu}{kT}\zeta(2) + \dots \right)
\end{aligned}$$

従って、

高温時 : $\mu \ll kT$

$$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{3}{4}\zeta(3) + \frac{\mu}{2kT}\zeta(2) + \dots\right) \quad (\theta = 1 \text{ for Fermi-Dirac})$$

$$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\zeta(3) + \frac{\mu}{kT}\zeta(2) + \dots\right) \quad (\theta = -1 \text{ for Bose-Einstein})$$

$$n\left(T, 0, \frac{\mu}{kT}\right) = \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(1 + \frac{\mu}{kT} + \dots\right) \quad (\theta = 0 \text{ for Maxwell-Boltzmann})$$

を得る。

【 $\mu/kT \gg 1$ のとき】

次に、 $\mu/kT \gg 1$ のとき、【Appendix 1:種々の特殊関数の公式—ツェータ関数】の

$$-Li_n(-e^\alpha) = \frac{\alpha^n}{n!} \quad (n=1), \quad \frac{\alpha^n}{n!} + \frac{\alpha^{n-2}}{(n-2)!}\zeta(2) + \dots \quad (n \geq 2)$$

を用いると、

$$\begin{aligned} n\left(T, 0, \frac{\mu}{kT}\right) &= -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(-e^{\mu/kT}) \quad (\theta = 1 \text{ for Fermi-Dirac}) \\ &= \frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 \left(\frac{(\mu/kT)^3}{3!} + \frac{(\mu/kT)^{3-2}}{(3-2)!}\zeta(2) + \dots\right) = \frac{g}{\pi^2} \left(\left(\frac{\mu}{\hbar}\right)^3 \frac{1}{3!} + \frac{\mu(kT)^2}{\hbar^3}\zeta(2) + \dots\right) \\ &= \frac{g}{6\pi^2} \left(\frac{\mu}{\hbar}\right)^3 \left(1 + 6\zeta(2)\left(\frac{kT}{\mu}\right)^2 + \dots\right) \end{aligned}$$

なので、

低温時 : $\mu \gg kT$

$$n\left(T, 0, \frac{\mu}{kT}\right) = -\frac{g}{\pi^2} \left(\frac{kT}{\hbar}\right)^3 Li_3(-e^{\mu/kT}) = \frac{g}{6\pi^2} \left(\frac{\mu}{\hbar}\right)^3 \left(1 + 6\zeta(2)\left(\frac{kT}{\mu}\right)^2 + \dots\right) \quad (\text{Fermi-Dirac})$$

がわかる。

Appendix 1:種々の特殊関数の公式

ベッセル関数

$$\int_x^\infty z(z^2 - x^2)^{\nu-1} e^{-az} dz = 2^{\nu-\frac{1}{2}} (\sqrt{\pi})^{-1} \alpha^{\frac{1}{2}-\nu} x^{\nu+\frac{1}{2}} \Gamma(\nu) K_{\nu+\frac{1}{2}}(\alpha x) \quad (\alpha x > 0)$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{\nu^2 - 1/4}{2x} + \frac{(\nu^2 - 1/4)(\nu^2 - 9/4)}{8x^2} + \dots\right) \quad (x \gg 1)$$

$$K_\nu(x) = \sum_{k=0}^{\nu-1} (-1)^k \frac{(\nu-k-1)!}{k!} \left(\frac{1}{2}x\right)^{2k-\nu} + (-1)^{\nu+1} \sum_{k=0}^\infty \frac{\left(\frac{1}{2}x\right)^{2k+\nu}}{k!(\nu+k)!} \left[\ln \frac{1}{2}x + \ln \overset{\text{Euler's cst}}{0.5772} \gamma + \sum_{i=1}^k \frac{1}{2i} + \sum_{i=1}^{k+\nu} \frac{1}{2i} \right] \quad (x < 1)$$

$$K_2(z) = \frac{2}{z}K_1(z) + K_0(z), \quad z \frac{d}{dz}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$

ツェータ関数

$$\int_0^\infty \frac{z^{\nu-1}}{e^{\alpha z} + 1} dz = \frac{1}{\alpha^\nu} (1 - 2^{1-\nu}) \Gamma(\nu) \zeta(\nu), \quad \int_0^\infty \frac{z^{\nu-1}}{e^{\alpha z} - 1} dz = \frac{1}{\alpha^\nu} \Gamma(\nu) \zeta(\nu)$$

$$\zeta(2m) = (-1)^{m-1} \frac{(2\pi)^{2m}}{2} \frac{B_{2m}}{(2m!)}, \quad \zeta(2m-1) = -\frac{B_{2m}}{2m}, \quad \zeta(\nu) = \sum_{n=1}^\infty \frac{1}{n^\nu} \quad (\nu > 1)$$

$$B_{2m} : \text{Bernoulli数} \quad \sum_{m=0}^\infty B_m \frac{t^m}{m!} = \frac{te^t}{e^t - 1} \quad \left(B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730} \right)$$

多重対数関数

$$\int_0^\infty \frac{x^{n-1}}{e^x - z} dx = \frac{Li_n(z) \Gamma(n)}{z} \quad (= \Gamma(n) \text{ at } z = 0)$$

$$Li_n(z) = \int_0^z \frac{Li_{n-1}(t)}{t} dt = \sum_{k=1}^\infty \frac{z^k}{k^n} = \frac{(-1)^n}{\Gamma(n-1)} \int_0^1 \frac{\ln^{n-2} t \ln(1-xt)}{t} dt \quad (n \geq 2)$$

$$Li_n(z^2) = 2^{n-1} [Li_n(z) + Li_n(-z)],$$

$$Li_1(z) = -\ln(1-z) = \int_0^z \frac{dt}{1-t}, \quad Li_n(1) = \zeta(n), \quad Li_n(-1) = (2^{1-n} - 1) \zeta(n)$$

For $0 < z \ll 1$,

$$Li_2(1-z) = \zeta(2) - Li_2(z) - \ln z \ln(1-z) = \zeta(2) - z(-\ln z + 1) + \dots$$

$$Li_3(1-z) = \zeta(3) - z\zeta(2) - \frac{1}{4}z^2(2\ln z + 2\zeta(2) - 3) + \dots$$

$$Li_n(1-z) = \zeta(n) - z\zeta(n-1) + \dots \quad (n > 2)$$

For $|z| \ll 1$,

$$Li_n(-1+z) = (2^{1-n} - 1)\zeta(n) + z(1 - 2^{2-n})\zeta(n-1) + \dots \quad (n > 2)$$

For $z \gg 1$,

$$Li_n(-z) = \frac{\ln^n z}{n!} + \dots \quad (n \leq 2), \quad -\frac{\ln^n z}{n!} - \frac{\ln^{n-2} z}{(n-2)!} \zeta(2) + \dots \quad (n > 2)$$

多重対数関数の詳細

定義は、

$$Li_n(z) = \frac{z}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^x - z} dx = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx \quad (z > 0)$$

である。

$$z < 1$$

$$\begin{aligned}
 Li_n(z) &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx = \frac{1}{\Gamma(n)} \int_0^\infty \frac{ze^{-x}x^{n-1}}{1 - ze^{-x}} dx = \frac{1}{\Gamma(n)} \int_0^\infty \left[ze^{-x} \sum_{k=0}^\infty (ze^{-x})^k \right] x^{n-1} dx \\
 &= \frac{1}{\Gamma(n)} \int_0^\infty \left[\sum_{k=0}^\infty (z^{-1}e^x)^{k+1} \right] x^{n-1} dx = \frac{1}{\Gamma(n)} \int_0^\infty \left[\sum_{k=1}^\infty (ze^{-x})^k \right] x^{n-1} dx = \frac{1}{\Gamma(n)} \sum_{k=1}^\infty z^k \int_0^\infty x^{n-1} e^{-kx} dx \\
 &= \frac{1}{\Gamma(n)} \sum_{k=1}^\infty z^k \int_0^\infty \left(\frac{kx}{k} \right)^{n-1} e^{-kx} d(kx) = \frac{1}{\Gamma(n)} \sum_{k=1}^\infty \frac{z^k}{k^{n-1}} \int_0^\infty y^{n-1} e^{-y} dy \Leftarrow \Gamma(n) = \int_0^\infty y^{n-1} e^{-y} dy \\
 &= \frac{1}{\Gamma(n)} \sum_{k=1}^\infty \frac{z^k}{k^{n-1}} \Gamma(n) = \sum_{k=1}^\infty \frac{z^k}{k^{n-1}} \\
 Li_n(-z) &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{-z^{-1}e^x - 1} dx = -\frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1} dx = \sum_{k=1}^\infty \frac{(-z)^k}{k^{n-1}}
 \end{aligned}$$

なので、

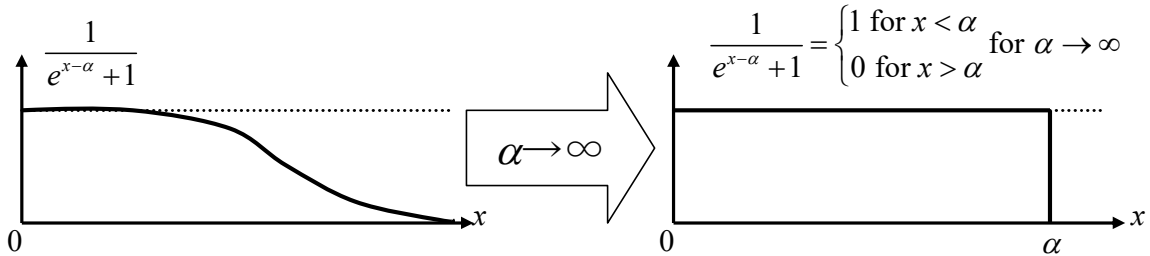
$$\boxed{\frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx = Li_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^{n-1}}, \quad \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1} dx = -Li_n(-z) = -\sum_{k=1}^\infty \frac{(-z)^k}{k^{n-1}} \quad (z < 1)}$$

を得る。

$$z \gg 1$$

$$-Li_n(-z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1} dx = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-\alpha} + 1} dx \Leftarrow z = e^\alpha$$

この関数の $z \gg 1 (\alpha \rightarrow \infty)$ の振る舞いは、「電子ガスの縮退」を記述する。このとき、 $\alpha \rightarrow \infty$ では、次のような置き換えを使用している。それは、下図：



のようなステップ関数である。つまり、 $\alpha \rightarrow \infty$ では、

$$\frac{1}{e^{x-\alpha} + 1} = \Theta(\alpha - x)$$

である。そこで、

$$\begin{aligned}
 -Li_n(-e^\alpha) &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-\alpha} + 1} dx = \frac{1}{\Gamma(n)} \int_0^\infty \left[\Theta(\alpha - x) + \left(\frac{1}{e^{x-\alpha} + 1} - \Theta(\alpha - x) \right) \right] x^{n-1} dx \\
 &= \frac{1}{\Gamma(n)} \int_0^\alpha x^{n-1} dx + \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{1}{e^{x-\alpha} + 1} - \Theta(\alpha - x) \right) x^{n-1} dx \\
 &= \frac{1}{\Gamma(n)} \int_0^\alpha x^{n-1} dx + \frac{1}{\Gamma(n)} \int_0^\alpha \left(\frac{1}{e^{x-\alpha} + 1} - 1 \right) x^{n-1} dx + \frac{1}{\Gamma(n)} \int_\alpha^\infty \frac{1}{e^{x-\alpha} + 1} x^{n-1} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(n)} \int_0^\alpha x^{n-1} dx + \frac{1}{\Gamma(n)} \int_0^\alpha \left(-\frac{e^{x-\alpha}}{e^{x-\alpha} + 1} \right) x^{n-1} dx + \frac{1}{\Gamma(n)} \int_\alpha^\infty \frac{1}{e^{x-\alpha} + 1} x^{n-1} dx \\
&= \frac{\alpha^n}{n\Gamma(n)} + \frac{1}{\Gamma(n)} \int_0^\alpha \left(-\frac{e^{x-\alpha}}{e^{x-\alpha} + 1} \right) x^{n-1} dx + \frac{1}{\Gamma(n)} \int_\alpha^\infty \frac{1}{e^{x-\alpha} + 1} x^{n-1} dx \\
&= \frac{\alpha^n}{\Gamma(n+1)} - \frac{1}{\Gamma(n)} \int_0^\alpha \frac{1}{1+e^{\alpha-x}} x^{n-1} dx + \frac{1}{\Gamma(n)} \int_\alpha^\infty \frac{1}{e^{x-\alpha} + 1} x^{n-1} dx \\
&\quad \leftarrow u = \alpha - x, v = x - \alpha \\
&= \frac{\alpha^n}{\Gamma(n+1)} - \frac{1}{\Gamma(n)} \int_\alpha^0 \frac{1}{1+e^u} (\alpha-u)^{n-1} (-du) + \frac{1}{\Gamma(n)} \int_0^\infty \frac{1}{e^v + 1} (v+\alpha)^{n-1} dv \\
&= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \int_{\alpha}^0 \frac{1}{1+e^u} (\alpha-u)^{n-1} du + \frac{1}{\Gamma(n)} \int_0^\infty \frac{1}{e^{\boxed{v}} + 1} (\boxed{v} + \alpha)^{n-1} d\boxed{v} \\
&\quad \leftarrow \boxed{\alpha \gg 1} \\
&= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \int_\infty^0 \frac{1}{1+e^u} (\alpha-u)^{n-1} du + \frac{1}{\Gamma(n)} \int_0^\infty \frac{1}{e^u + 1} (u+\alpha)^{n-1} du \\
&= \frac{\alpha^n}{\Gamma(n+1)} - \frac{1}{\Gamma(n)} \int_0^\infty \frac{\alpha^{n-1}}{1+e^u} \left(1 - \frac{u}{\alpha}\right)^{n-1} du + \frac{1}{\Gamma(n)} \int_0^\infty \frac{\alpha^{n-1}}{e^u + 1} \left(1 + \frac{u}{\alpha}\right)^{n-1} du \\
&= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \int_0^\infty \frac{\alpha^{n-1}}{1+e^u} \left[\left(1 + \frac{u}{\alpha}\right)^{n-1} - \left(1 - \frac{u}{\alpha}\right)^{n-1} \right] du \\
&= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \int_0^\infty \frac{\alpha^{n-1}}{1+e^u} \sum_{k=0}^{n-1} \left[{}_{n-1}C_k \left(\frac{u}{\alpha}\right)^k - {}_{n-1}C_k \left(-\frac{u}{\alpha}\right)^k \right] du \\
&= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \int_0^\infty \frac{\alpha^{n-1}}{1+e^u} \sum_{k=\text{odd}}^{n-1} 2 {}_{n-1}C_k \left(\frac{u}{\alpha}\right)^k du = \frac{\alpha^n}{\Gamma(n+1)} + 2 \frac{1}{\Gamma(n)} \sum_{k=\text{odd}}^{n-1} \frac{\alpha^{n-1}}{\alpha^k} {}_{n-1}C_k \int_0^\infty \frac{u^k}{1+e^u} du \\
&\quad \leftarrow \int_0^\infty \frac{u^{\nu-1}}{e^{\alpha u} + 1} du = \frac{1}{\alpha^\nu} (1-2^{1-\nu}) \Gamma(\nu) \zeta(\nu) \\
&= \frac{\alpha^n}{\Gamma(n+1)} + 2 \frac{1}{\Gamma(n)} \sum_{k=\text{odd}}^{n-1} \alpha^{n-1-k} {}_{n-1}C_k \frac{1}{\alpha^{k+1}} (1-2^{1-(k+1)}) \Gamma(k+1) \zeta(k+1) \\
&= \frac{\alpha^n}{\Gamma(n+1)} + 2 \frac{1}{\Gamma(n)} \sum_{k=\text{odd}}^{n-1} \alpha^{n-1-k} {}_{n-1}C_k (1-2^{-k}) \Gamma(k+1) \zeta(k+1)
\end{aligned}$$

従って、

$$\boxed{(\alpha \gg 1) -Li_n(-e^\alpha) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-\alpha} + 1} dx = \frac{\alpha^n}{\Gamma(n+1)} + 2 \frac{1}{\Gamma(n)} \sum_{k=\text{odd}}^{n-1} \alpha^{n-1-k} {}_{n-1}C_k (1-2^{-k}) \Gamma(k+1) \zeta(k+1)}$$

を得る。最初の2項くらいは・・・ $n=1$ のとき：

$$-Li_1(-e^\alpha) = \frac{1}{\Gamma(1)} \int_0^\infty \frac{x^{1-1}}{e^{x-\alpha} + 1} dx = \frac{\alpha}{\Gamma(2)} = \alpha$$

$n \geq 2$ のとき：

$$\begin{aligned}
 -Li_{n \geq 2}(-e^\alpha) &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^{x-\alpha} + 1} dx = \frac{\alpha^n}{\Gamma(n+1)} + \cancel{\frac{1}{\Gamma(n)} \alpha^{n-2} {}_{n-1}C_1 (1-2^{-1}) \Gamma(2) \zeta(2) + \dots} \\
 &= \frac{\alpha^n}{\Gamma(n+1)} + \frac{1}{\Gamma(n)} \alpha^{n-2} (n-1) \zeta(2) + \dots = \frac{\alpha^n}{n!} + \frac{1}{(n-1)!} \alpha^{n-2} (n-1) \zeta(2) + \dots \\
 &= \frac{\alpha^n}{n!} + \frac{\alpha^{n-2}}{(n-2)!} \zeta(2) + \dots
 \end{aligned}$$

なので、まとめて、

$$(\alpha \gg 1) \quad -Li_n(-e^\alpha) = \begin{cases} \frac{\alpha^n}{n!} & (n=1) \\ \frac{\alpha^n}{n!} + \frac{\alpha^{n-2}}{(n-2)!} \zeta(2) + \dots & (n \geq 2) \end{cases}$$

を得る。